Integration by parts for locally smooth laws

and applications to

jump type stochastic equations

Vlad Bally and Emmanuelle Clément

Université de Marne-la-Vallée

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1 Jump type stochastic equations

$$X_t = x + \int_0^t \int_E c(s,a,X_{s-}) d\widetilde{N}(s,a) + \int_0^t g(s,X_s) ds,$$

$$d\widetilde{N}(s,a) = dN(s,a) - ds\mu(a) \quad \text{Poisson point measure.}$$

PROBLEM : Give sufficient conditions in order that :

◊ $P_{X_t}(dy) = p_t(x,y)dy.$

◊ Regularity of $p_t(x,y)$.

◊ Behavior as $|x-y| \to \infty$.

Approaches :

1. Malliavin calculus with respect to the **jump amplitudes**.

2. Malliavin calculus with respect to the **jump times**.

Discretization

$$\begin{aligned} X_t^n &= x + \int_0^t \int_{E_n} c(s, a, X_{s-}^n) d\widetilde{N}(s, a) + \int_0^t g(s, X_s^n) ds \\ &= x + \sum_{k=1}^{\infty} c(T_k^n, \Delta_k^n, X_{T_k^n -}^n) 1_{[0,t]}(T_k^n) + \int_0^t g_n(s, X_s^n) ds. \end{aligned}$$

Difficulty :The coefficients are discontinuous with respect to T_k^n.

2. Malliavin calculus with respect to the **jump amplitudes.** We look to the equation

$$X_t^n = x + \int_0^t \int_{E_n \times R_+} c(a, X_{s-}^n) 1_{\{u<\gamma(a,X_{s-}^n)\}} d\widetilde{N}(s,a,u) + \int_0^t g(X_s^n) ds$$
$$= x + \sum_{T_k^n \leq t}^{\infty} c(\Delta_k^n, X_{T_k^n-}^n) 1_{\{U_k<\gamma(\Delta_k^n, X_{T_k^n-}^n)\}} + \int_0^t g_n(X_s^n) ds$$

with intensity measure

$$\mu(da, du) = \phi(a) da \times 1_{[0,\infty)}(u) du.$$

Motivation : Infinitesimal operator

$$L\psi(x) = g\nabla\psi(x) + \int_E (\psi(x+c(a,x)) - \psi(x)) K(x, da), \quad K(x,da) = \phi(a)\gamma(x,a) da.$$

Difficulty :The coefficients are discontinues with respect to $\Delta_k^n, X_{T_k^n-}^n$.

Theorem (Malliavin Caculus for jump times)

$$P_{X_t}(dy) = p_t(y)dy$$

Equation :

$$X_t = x + \int_0^t \int_E c(s, a, X_{s-})d\widetilde{N}(s, a) + \int_0^t g(s, X_s)ds$$

Hypothesis

$(\mathbf{H}_1)$

$$|c(t, a, x)| + \sum_{k,p} \left|\partial_x^k \partial_t^p c(t, a, x)\right| \leq \overline{c}(a) \quad \int \overline{c}(a)d\mu(a) < \infty,$$
$$|b(t, x)| + \sum_{k,p} \left|\partial_x^k \partial_t^p b(t, x)\right| \leq C$$

(H_2)

$$\frac{|\partial_x c(t, a, x)|}{|1 + c(t, a, x)|} \leq \widehat{c}(a) \quad \int \widehat{c}(a)d\mu(a) < \infty$$

(H_3) **Non Degeneracy**

$$\alpha(t,a,x) := g(t,x) - g(t,x+c(t,a,x)) + (g\partial_x c + \partial_t c)(t,a,x) \geq \underline{\alpha}(a) > 0$$

We suppose that

$$\underline{\lim}_{n\to\infty} \frac{1}{\mu(E_n)} \ln \int_{E_n} \frac{1}{\underline{\alpha}(a)} d\mu(a) = \theta < \infty, \quad E_n \uparrow E.$$

Case 1.

$$\theta = 0 \quad \Longrightarrow \quad p_t(y) \in C^\infty$$

Case 2.

$$\theta > 0, \quad t \geq \theta(k+2)^3 \quad \Longrightarrow \quad p_t(y) \in C^k.$$

Theorem (Malliavin Caculus for jump amplitudes) Equation :

$$X_t = x + \int_0^t \int_E c(s, a, X_{s-}) 1_{\{u \leq \gamma(a, X_{s-})\}} d\widetilde{N}(s, a) + \int_0^t g(s, X_s) ds$$

With

$$\mu(da) = \phi(a) da.$$

Hypothesis : $(H_1), (H_2)$

(H_3)

$$\ln \phi \in C_1^b(R^d).$$

(H_4) **Non-Degeneracy**

$$(\nabla_a c)(\nabla_a c)^*(a, x) \geq \underline{c}(a), \quad \overline{\gamma}(a) \geq \gamma(a, x) \geq \underline{\gamma}(a)$$

and

$$\underline{\lim}_{M \to \infty} \int_{\{\underline{c} \geq M\}} \underline{\gamma}(a) d\mu(a) = \theta.$$

(H_5)

$$\int \left| \partial_x^k \ln \gamma(a,x) \right| \overline{\gamma}(a) d\mu(a) < \infty$$

Then :

Case 1.

$$\theta = \infty \quad \Longrightarrow \quad p_t \in C^\infty$$

Case 2.

$$\theta < \infty \quad \Longrightarrow \quad p_t \in C^k \quad if \quad t \geq \frac{12d(k+d)}{\theta}.$$

2 Finite dimensional Malliaivn calculus (B,M.P.Bavouzet, M.Mesaoud, E. Clemant)

We consider a multidimensional random variable $V = (V_1, ..., V_m)$ and $a_i(\omega) < b_i(\omega), i = 1, ..., m$. We also consider a $\sigma-$algebra G (the noise which is not employed in the calculus).

Hypothesis. There exists $p(\omega, y)$ such that

$$E_G(\phi(V)) = \int_{R^m} \phi(y) p(\omega, y) dy$$

and

$$y_i \to p(\omega, y) 1_{(a_i, b_i)}(y_i) \quad smooth, \quad \omega \to p(\omega, y) \quad G - measurable.$$

Simple functionals : $F = f(\omega, V_1, ..., V_m)$ $f(\omega, \circ)$ smooth on (a_i, b_i),

Derivatives : $D_p F = 1_{(a_p, b_p)}(V_p)\partial_{V_p} f(V_1, ..., V_m), \quad p = 1, ..., m$

Simple Processes : $U = (U_p)_{p=1,...,m}$ $\qquad U_p = u_p(V_1, ..., V_m)$

Weights : $\pi_p(\omega, y)$ which is G measurable in ω, smooth in y_p and null on $(a_p, b_p)^c$.

Scalar product :

$$\langle U, V \rangle_\pi := \sum_{p=1}^{m} U_p \times V_p \times \pi_p$$

Malliaivn covariance matrix : Let $F = (F^1, ..., F^d)$.

$$\sigma_F^{ij} = \left\langle DF^i, DF^j \right\rangle_\pi, \quad i, j = 1, ..., m.$$

Duality Formula

$$E(\langle DF, U\rangle_\pi) = E(F\delta_\pi(U))$$

with

Divergence operator

$$\delta_\pi(U) = -\sum_{p=1}^{m} (\partial_{V_p}(U_p\pi_p) + (U_p\pi_p) \times \partial_{V_p} \ln p(\omega, V_p))$$

Proof.

$$E(\langle DF, U\rangle_\pi) = \sum_{p=1}^{m} E(E_G(D_pF \times U_p \times \pi_p)).$$

And

$$\begin{aligned} E_G(D_pF \times U_p \times \pi_p) &= \int_{R^{m-1}} \int_{a_p}^{b_p} \partial_{y_p} f(y) u_p(y) \pi_p(y) p(y) dy \\ &= -\int_{R^{m-1}} \int_{a_p}^{b_p} f(y)(\partial_{y_p}(u_p\pi_p)(y)p(y) + (u_p\pi_p)(y)\partial_{y_p} p(y) dy \end{aligned}$$

The second term is equal to

$$\int_{R^{m-1}} \int_{a_p}^{b_p} f(\ldots, y)(\partial_{y_p}(u_p\pi_p)(y) + (u_p\pi_p)(y)\frac{\partial_{y_p} p(y)}{p(y)}) p(y) dy$$
$$= E_{G_p}(F(\partial_{V_p}(U_p\pi_p) + (U_p\pi_p)\partial_{V_p} \ln p(V_p)).$$

We take the sum....□

Integration by parts formula Let $\phi : R^d \to R$

$$E(\partial_i \phi(F) G) = E(\phi(F) H_i(F,G)) \quad with$$

$$H_i(F,G) = \sum_{j=1}^{d} \delta_\pi (G \gamma_F^{ij} DF^j) \quad and \quad \gamma_F = \sigma_F^{-1}.$$

Proof. 1. Chain rule + Duality. □

Baisic estimate

$$E\left|H_i(F,G)\right|^p \leq C \left\| \det \sigma_F^{-1} \right\|_{p'} \|F\|_{2,p'} \|G\|_{1,p'} \times \|\partial \ln p\|_{p'}$$

with

$$\|F\|_{2,p'} = \|F\|_{p'} + \|DF\|_{p'} + \left\| D^2 F \right\|_{p'}.$$

Duality Formula with Border terms

$$E_G(D_pF \times U_p \times \pi_p) = \int_{a_p}^{b_p} \partial_{y_p} f(y) u_p(y) \pi_p(y) p(y) dy$$
$$= - \int_{a_p}^{b_p} f(y) (\partial_{y_p}(u_p \pi_p)(y) p(y) + (u_p \pi_p)(y) \partial_{y_p} p(y) dy$$
$$+ f(b_p) u_p(b_p) \pi_p(b_p) p(b_p) - f(a_p) u_p(a_p) \pi_p(a_p) p(a_p)$$

Then the **duality** reads

$$E(\langle DF, U \rangle_\pi) = E(F \delta_\pi(U)) + E[F, Up]$$

And **Integration by Parts**

$$E(\phi'(F)G) = E(\phi(F) H(F, G)) + E(B(\phi, F, G))$$

3 Passage to the limit

Malliavin Calculus : Infinite dimensional differential calculus :

$$F_n \to F \quad in \quad L^2(\Omega),$$
$$DF_n \to DF \quad in \quad L^2(R_+ \times \Omega),$$
$$\delta(DF_n) = LF_n \to LF \quad in \quad L^2(\Omega).$$

Then we have the Integration by parts formula

$$E(\phi'(F)G) = E(\phi(F)H(F,G))$$

and we iterate :

$$E(\phi^{(k)}(F)G) = E(\phi(F)H^k(F,G)).$$

Density of the law :

$$\left|E(\phi^{(k)}(F))\right| = \left|E(\phi(F)H^k(F,1))\right| \leq \|\phi\|_\infty E(\left|H^k(F,1)\right|)$$

and this implies

$$P_F(dy) = p_F(y)dy \quad and \quad p_F \in C^k.$$

Alternative approach :

$$\left|E(\phi^{(k)}(F))\right| = \lim_n \left|E(\phi^{(k)}(F_n))\right| = \lim_n \left|E(\phi(F_n)H^k(F_n,1))\right|$$
$$\leq \|\phi\|_\infty \sup_n E(\left|H^k(F_n,1)\right|).$$

Conclusion : We need

$$a) \quad F_n \to F \quad \text{in law (not in } L^2\text{)},$$
$$b) \quad \sup_n E(\left|H^k(F_n,1)\right|) < \infty.$$

Question : What happens if

$$\sup_n E(\left|H^k(F_n, 1)\right|) = \infty.$$

Example : in the previous equation we have

$$E(\delta(DX_t^n)) \sim n \uparrow \infty.$$

Consequence : We can not define $\delta(DX_t)$!

Example

$$X_t^n = x + \int_0^t \int_{E_n \times R_+} c(a, X_{s-}^n) 1_{\{u<\gamma(a, X_{s-}^n)\}} d\widetilde{N}(s,a,u) + \int_0^t g(X_s^n) ds$$

$$\mu(da, du) = \phi(a) da \times du.$$

We have

$$E(f^{(k)}(X_t^n)) = E(f(X_t^n) H_k^n) \quad with$$

$$\sup_n E(|H_k^n|^p) = \infty \quad \text{because} \quad \delta(DX_t^n) \uparrow \infty.$$

Consequence : We can not pass to the limit.

Approach using the Fourier transform. We have to prove that

$$\int_R |\xi|^p |\widehat{p}(\xi)|\, d\xi < \infty \quad with \quad \widehat{p}(\xi) = E(\exp(i\xi X_t)).$$

We denote

$$\widehat{p}_n(\xi) = E(\exp(i\xi X_t^n)), \quad \varepsilon_n = E\,|X_t - X_t^n|\,.$$

And we write

$$|\widehat{p}(\xi)| \le |\widehat{p}(\xi) - \widehat{p}_n(\xi)| + |\widehat{p}_n(\xi)| \le |\xi|\,\varepsilon_n + \frac{1}{|\xi|^k} E(\partial_x^k \exp(i\xi X_t^n))$$

$$\le |\xi|\,\varepsilon_n + \frac{1}{|\xi|^k} E(|H_k^n|).$$

Suppose that

$$\varepsilon_n = \frac{1}{n} \quad and \quad E(|H_k^n|) = C_k n^2.$$

Then

$$|\widehat{p}(\xi)| \le \frac{|\xi|}{n} + C_k \frac{n^2}{|\xi|^k} \quad \forall \xi \in N.$$

Equilibrium between n and ξ : We take

$$n = |\xi|^{\rho}$$

Then

$$|\widehat{p}(\xi)| \leq |\xi|^{1-\rho} + C_k \frac{1}{|\xi|^{k-2\rho}} \quad \forall \xi, n \in N.$$

We want

$$1 - \rho = -(k - 2\rho) \quad \Leftrightarrow \rho = \frac{k+1}{3}$$

Conclusion : if we choose

$$n(\xi) = |\xi|^{\frac{k+1}{3}}$$

then we obtain

$$|\widehat{p}(\xi)| \leq |\xi|^{\frac{-k+2}{3}}$$

so that

$$\int_R |\xi|^p \, |\widehat{p}(\xi)| \, d\xi < \infty \quad \forall p \in N.$$

Example

$$X^n_{T_{k+1}} = X^n_{T_k-} + 1_{B_n}(\Delta_k)c(\Delta_k, X^n_{T_k-})1_{\{0\leq U_k\leq\gamma(\Delta_k, X^n_{T_k-})\}} + \int_{T_k}^{T_{k+1}} g(X^n_s)ds.$$

We have

$$X^n_{T_{k+1}-} = F(T_1, ..., T_{k+1}, U_1, ..., U_{k+1}, \Delta_1, ..., \Delta_k).$$

We have

$$G = \sigma(T_i, i \in N, U_i, i \in N), \quad \Delta_i \sim \phi(z)dz.$$

Alternative representation. We construct a Markov chain which has the same law as $X^n_{T_k}$: take

$$P(\overline{\Delta}_k = dz \mid \overline{X}^n_{T_k-} = x) = 1_{B_n}(\Delta_k)\gamma(z, \overline{X}^n_{T_k-})\phi(z)dz + 1_{B^c_{n+2}}(\Delta_k)\theta(x) \times \delta_{\Delta_*}.$$

We construct by recurrence

$$\overline{X}^n_{T_{k+1}} = \overline{X}^n_{T_k-} + c(\overline{\Delta}_k, \overline{X}^n_{T_k-}) + \int_{T_k}^{T_{k+1}} g(\overline{X}^n_s)ds.$$

We have $\overline{X}^n_{T_{k+1}} \sim X^n_{T_{k+1}}$

We work with

$$\overline{X}^n_{T_{k+1}} = \overline{F}(T_1, ..., T_{k+1}, U_1, ..., U_{k+1}, \overline{\Delta}_1, ..., \overline{\Delta}_k).$$

Weights

$$\pi_k = \Phi_n(\overline{\Delta}_k) \quad with \quad 1_{B_{n+1}}(\overline{\Delta}_k) \leq \Phi_n(\overline{\Delta}_k) \leq 1_{B_{n+2}}(\overline{\Delta}_k)$$

Conditional density

$$P(\overline{\Delta}_k \in dz \mid \overline{X}^n_{T_k-})\Phi_n(z) = p_k(\omega, z)\Phi_n(z) = \Phi_n(z)\gamma(z, \overline{X}^n_{T_k-})\phi(z)dz$$

$$p(\omega, z_1, ..., z_k) = \prod_{i=1}^{k} p_k(\omega, z_i)$$

Conclusion : We have now :

☐ A (locally) smooth conditional density,

☐ A smooth functional.

Divergence operator - Logarithmic derivative of the density.. :

$$
\begin{aligned}
\partial_{z_i} \ln p(\omega, z_1, ..., z_k) &= \partial_{z_i} \ln \prod_{j=1}^{k} p_j(\omega, z_j) \\
&= \sum_{j=1}^{k} \partial_{z_i} \ln \Phi_n(z_j)\gamma(z_j, \overline{X}^n_{T_j-})\phi(z_j) \\
&= \partial_{z_i} \ln \Phi_n(z_i)\gamma(z_i, \overline{X}^n_{T_i-}) + \sum_{j=1}^{k} \partial_x \ln \gamma(z_j, \overline{X}^n_{T_j-}) \times \partial_{z_i} \overline{X}^n_{T_j-}
\end{aligned}
$$

Problem : The perturbation of z_k propagates to $X^n_{T_j}, j \geq k+1$.

Consequence : The divergence operator blows up..

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