Dirichlet forms applied to Poisson measures : simplified construction and the double Fock space

Nicolas BOULEAU

common work with Laurent DENIS

We develop the Malliavin calculus for Poisson measures in the BGJ way of acting on the jumps size, but with general local Dirichlet forms. The construction that we present replace the Friedrichs' method by a Monte Carlo argument. The choice of the gradient as a marked point process yields a very simple structure in terms of chaos and Fock spaces. We give a collection of remarkable formulas including the lent particle formulas for the gradient, the generator and the "divergence". We begin to study the celebrated functional inequalities in this framework.

I. Some results yielded by the method

Example 1) Let us consider the following SDE driven by a two dimensional Brownian motion

$$\left\{\begin{array}{ll} X_t^1 & = z_1 + \int_0^t dB_s^1 \\ X_t^2 & = z_2 + \int_0^t 2X_s^1 dB_s^1 + \int_0^t dB_s^2 \\ X_t^3 & = z_3 + \int_0^t X_s^1 dB_s^1 + 2\int_0^t dB_s^2. \end{array}\right. \tag{1}$$

This diffusion is degenerate and Hörmander conditions are not fulfilled : the diffusion remains on the quadric of equation $\frac{3}{4}x_1^2 - x_2 + \frac{1}{2}x_3 - \frac{3}{4}t = C$. Let us now consider the same equation driven by a Lévy process :

$$\left\{\begin{array}{ll} Z_t^1 & = z_1 + \int_0^t dY_s^1 \\ Z_t^2 & = z_2 + \int_0^t 2Z_{s_-}^1 dY_s^1 + \int_0^t dY_s^2 \\ Z_t^3 & = z_3 + \int_0^t Z_{s_-}^1 dY_s^1 + 2\int_0^t dY_s^2 \end{array}\right.$$

Under the hypotheses of the method : the Lévy measure σ of (Y^1, Y^2) is such that $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2), \sigma, \mathcal{C}_k^\infty, \gamma)$ with the carré du champ operator $\gamma[f] = y_1^2 f_1'^2 + y_2^2 f_2'^2$ is closable and $\mathcal{D}(a) \subset \mathcal{C}_k^\infty$ and natural hypotheses yielding the EID property,

easy application of the *lent particle formula* yields that Z_t has a density on $\mathbb{R}^3$ as soon as the Lévy measures of Y^1 and Y^2 are infinite.

No minoration is supposed of the growth of the Lévy measure near 0 as assumed by many authors.

Example 2) Density for the multiple Wiener-Ito integrals $(I_1(g), \ldots, I_n(g^{\otimes n}))$.

Contrarily to the Wiener case the random variables $I_m(g)$ are not regular in general. (their distribution may contain Dirac masses)

Under the hypotheses of the method : (BC) hypothesis on the bottom space and (EID) on the upper space, we have :

For $g \in L^\infty \cap \mathbf{d}$ such that $\nu\{\gamma[g] > 0\} = +\infty$ the vector $(I_1(g), \ldots, I_n(g^{\otimes n}))$ has a density on $\mathbb{R}^n$.

This result is quite different from what happens on the Wiener space since there the law of $(I_1(f), \ldots, I_n(f^{\otimes n}))$ is carried by the algebraic curve of equation

$$\begin{cases} x_2 = 2!H_2(\|f\|^2, x_1) \\ \vdots \\ x_n = n!H_n(\|f\|^2, x_1) \end{cases}$$

where $H_n(\lambda, x)$ is the Hermite polynomial given by $e^{tx - \frac{t^2\lambda}{2}} = \sum_{n=0}^{\infty} t^n H_n(\lambda, x)$.

Example 3) Gas of Brownian particles.

We consider a gas of Brownian particles in $\mathbb{R}^3$. Each particle is independent, the initial positions are distributed in $\mathbb{R}^3$ along a Poisson measure with uniform intensity.

We study the lowest distance of a particle to the origin during the time interval $[0,1]$.

The bottom space is $(X, \mathcal{X}, \nu) = (\mathbb{R}^3 \times W, \mathcal{B}(\mathbb{R}^3) \times \mathcal{W}, \lambda^3 \times m)$ where λ^3 is the 3-dimensional Lebesgue measure, that we equip with the product Dirichlet structure of the zero form on $\mathbb{R}^3$ and the O-U-form on the Wiener space. The structure $(X, \mathcal{X}, \nu, \mathbf{d}, \gamma)$ is thus naturally endowed with a gradient with values in $L^2(\hat{m})$. The hypothesis (BC) is fulfilled.

Using the lent particle method and an argument developped by Nualart-Vivès we obtain that the functional

$$H(\omega) = \inf_{\substack{t \in [0,1] \\ (x, \mathrm{w}) \in \operatorname{supp} \omega}} |x + B_t(\mathrm{w})|.$$

has a density.

II. Construction

II.1. Poisson multiple integrals and chaos (without Dirichlet forms)

In 1951 Kyiosi Ito [28] gave a rigorous definition of multiple integrals in the case of Gaussian measures but in a way which transposes naturally to the Poisson case:

$(X, \mathcal{X}, \nu)$ is a measured space, ν is continuous, $\nu^{\times n}$ doesn't charge the diagonal sets $\{(x_1, \ldots, x_n) : \exists i \neq j \; x_i = x_j\}$.

A real function f defined on $(X, \mathcal{X}, \nu)^n$ is said *elementary* if it is finite linear combination of indicator functions of finitely measured product sets and if it vanishes on diagonals.

If $A_1, A_2, \ldots, A_k$ are the product sets involved in the definition, f may be written

$$f(x_1, \ldots, x_n) = \sum a_{i_1,\cdots,i_n} 1_{A_{i_1}}(x_1) \cdots 1_{A_{i_n}}(x_n)$$

where the $A_1, A_2, \ldots, A_k$ are disjoint, of finite measure, and where $a_{i_1,\cdots,i_n}$ vanishes if two indices $i_1, \ldots, i_n$ are equal.

For such an elementary function we put

$$I_n(f) = \sum a_{i_1,\cdots,i_n} \tilde{N}(A_{i_1}) \cdots \tilde{N}(A_{i_n})$$

I_n is linear and satisfy

$$I_n(f) = I_n(\tilde{f}) \tag{2}$$

where $\tilde{f} = \frac{1}{n!} \sum_{\sigma \in \mathcal{S}} f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$ is the symmetrized of f.

It holds

$$\langle I_n(f), I_m(g) \rangle_{L^2(\mathbb{P})} = \delta_{m,n} \, n! \langle \tilde{f}, \tilde{g} \rangle_{L^2(\nu)} \tag{3}$$

Indeed, the sets $A_1, A_2, \ldots, A_k$ may be supposed to be the same for f and g, let us suppose first $m = n$:

$$\langle I_n(f), I_n(g)\rangle =$$

$$\langle \sum_{i_1<\cdots<i_p} (\sum_{\sigma\in\mathcal{S}} a_{\sigma(i_1)\cdots\sigma(i_n)})\tilde{N}(A_{i_1})\cdots\tilde{N}(A_{i_n}) \ , \ \sum_{j_1<\cdots<j_p} (\sum_{\tau\in\mathcal{S}} b_{\tau(j_1)\cdots\tau(j_n)})\tilde{N}(A_{j_1})\cdots\tilde{N}(A_{j_n})\rangle$$

$$= \sum_{i_1<\cdots<i_p} (\sum_{\sigma\in\mathcal{S}} a_{\sigma(i_1)\cdots\sigma(i_n)})(\sum_{\tau\in\mathcal{S}} b_{\tau(i_1)\cdots\tau(i_n)})\nu(A_{i_1})\cdots\nu(A_{i_n})$$

$$= \frac{1}{n!}\sum_{i_1,\cdots,i_p} (\sum_{\sigma\in\mathcal{S}} a_{\sigma(i_1)\cdots\sigma(i_n)})(\sum_{\tau\in\mathcal{S}} b_{\tau(i_1)\cdots\tau(i_n)})\nu(A_{i_1})\cdots\nu(A_{i_n})$$

$$= n!\sum_{i_1,\cdots,i_p} (\frac{1}{n!}\sum_{\sigma\in\mathcal{S}} a_{\sigma(i_1)\cdots\sigma(i_n)})(\frac{1}{n!}\sum_{\tau\in\mathcal{S}} b_{\tau(i_1)\cdots\tau(i_n)})\nu(A_{i_1})\cdots\nu(A_{i_n}) = n!\langle\tilde{f},\tilde{g}\rangle.$$

This last equality is due to

$$\tilde{f} = \frac{1}{n!}\sum_{\sigma\in\mathcal{S}}\sum_{i_1\cdots i_n} a_{i_1\cdots i_n}1_{A_{i_1}}(x_{\sigma(1)})\cdots 1_{A_{i_n}}(x_{\sigma(n)})$$

$$= \sum_{i_1\cdots i_n}\frac{1}{n!}\sum_{\sigma\in\mathcal{S}} a_{i_1\cdots i_n}1_{A_{i_1}}(x_{\sigma(1)})\cdots 1_{A_{i_n}}(x_{\sigma(n)}) = \sum_{i_1\cdots i_n}\frac{1}{n!}\sum_{\tau\in\mathcal{S}} a_{\tau(i_1)\cdots\tau(i_n)}1_{A_{\tau(i_1)}}(x_1)\cdots 1_{A_{\tau(i_n)}}(x_n)$$

and hence

$$\nu^{\times n}(\tilde{f}\tilde{g}) = \sum_{i_1\cdots i_n}\frac{1}{n!}\sum_{\tau} a_{\tau(i_1)\cdots\tau(i_n)}\frac{1}{n!}\sum_{\tau} b_{\tau(i_1)\cdots\tau(i_n)}\nu^{\times n}(A_{\tau(i_1)}\times\cdots\times A_{\tau(i_n)})$$

taking in account that $\nu^{\times n}(A_{\sigma(i_1)}\times\cdots\times A_{\sigma(i_n)}) = \nu(A_{i_1})\cdots\nu(A_{i_n})$.

If $m \neq n$ all product have expectation zero.

Then from (3)

$$\|I_n(f)\|^2_{L^2} = n!\|\tilde{f}\|^2_{L^2} \leqslant n!\|f\|^2_{L^2}$$

and I_n extends by density to $f \in L^2(\nu^{\times n})$ and we have $\forall f \in L^2(\nu^{\times n})$, $\forall g \in L^2(\nu^{\times m})$

$$I_n(f) = I_n(\tilde{f}) \qquad \langle I_n(f), I_m(g)\rangle_{L^2(\mathbb{P})} = \delta_{m,n}\, n!\langle \tilde{f}, \tilde{g}\rangle_{L^2(\nu)}. \tag{4}$$

The subvector space of $L^2(\Omega, \mathcal{A}, \mathbb{P})$ generated by the $I_n(f)$, $f \in L^2(X^n, \mathcal{X}^{\otimes n}, \nu^{\times n})$ is the Poisson chaos of order n denoted C_n. the equality

$$L^2(\Omega, \mathcal{A}, \mathbb{P}) = \mathbb{R} \bigoplus_{n=1}^{+\infty} C_n. \tag{5}$$

has been proved by K. Ito (cf [29]) in 1956 using that the set $\{N(E_1) \cdots N(E_k)$, (E_i) disjoint in $\mathcal{X}\}$ is total in $L^2(\Omega, \mathcal{A}, \mathbb{P})$.

There are now many proofs of this result : with combinatorics of diagonals Rota-Wallstrom [44], with structure transportation on $\mathbb{R}_+$ (cf Dellacherie-Maisonneuve-Meyer [19] p207).

Thanks to the density of the chaos one gets the following expansion (cf Surgailis [50]) for $u \in L^1 \cap L^\infty(\nu)$ s.t. $\|u\|_\infty < 1$,

$$e^{N(\log(1+u))-\nu(u)} = 1 + \sum_{n=1}^{+\infty} \frac{1}{n!} I_n(u^{\otimes n}). \tag{6}$$

II.2. Semigroups and closed forms

We suppose there is a local Dirichlet structure with OCC on the bottom space $(X, \mathcal{X}, \nu, \mathbf{d}, \gamma)$. Let p_t be the associated symmetric strongly continuous contraction semigroup on $L^2(\nu)$.

The product structure $(X, \mathcal{X}, \nu, \mathbf{d}, \gamma)^n$ will be denoted $(X^n, \mathcal{X}^{\otimes n}, \nu^{\times n}, \mathbf{d}_n, \gamma_n)$. It is equipped with the Dirichlet form $e_n[f] = \frac{1}{2}\int \gamma_n[f]d\nu$. The semigroup associated with e_n is denoted $p_t^{\otimes n}$, for $f \in L^2(\nu)$ it satisfies $p_t^{\otimes n}(f^{\otimes n}) = (p_t f)^{\otimes n}$. Choosing a gradient $\flat$ for the bottom structure induces a gradient on $(X^n, \mathcal{X}^{\otimes n}, \nu^{\times n}, \mathbf{d}_n, \gamma_n)$ denoted $(\cdot)^{\flat_n}$ with values in $(L^2_0(R, \mathcal{R}, \rho))^{\otimes n}$

$$\begin{aligned}(f^{\flat_n})(x_1, r_1, x_2, r_2, \cdots, x_n, r_n) =\\ (f(\cdot, x_2, \cdots, x_n))^\flat(x_1, r_1) + (f(x_1, \cdot, x_3, \cdots, x_n))^\flat(x_2, r_2) + \cdots\end{aligned} \tag{7}$$

If f is symmetric, $f^{\flat_n}$ is symmetric of the pairs (x_i, r_i).

For $u \in L^1 \cap L^\infty(\nu)$ with small norm $\|u\|_\infty$ we define P_t thanks to relation (6) by

$$P_t e^{N(\log(1+u))-\nu(u)} = 1 + \sum_{n=1}^{+\infty} \frac{1}{n!} I_n((p_t u)^{\otimes n}) \tag{8}$$

P_t is easily shown to be a symmetric strongly continuous contraction semigroup on $L^2(\mathbb{P})$ verifying for $F \in L^2(\mathbb{P})$ with chaos expansion $F = \mathbb{E}F + \sum_{n \geqslant 1} I_n(f_n)$

$$P_t F = \mathbb{E}F + \sum_{n \geqslant 1} I_n((p_t)^{\otimes n} f_n) \tag{9}$$

The closed form $\mathcal{E}$ associated with P_t and its domain are defined by

$$\mathbb{D} = \{F \in L^2(\mathbb{P}) : \lim_{t\downarrow 0} \uparrow \frac{1}{t}\langle F - P_t F, F\rangle_{\mathbb{P}} < +\infty)$$

$$\mathcal{E}[F] = \lim_{t\downarrow 0} \uparrow \frac{1}{t}\langle F - P_t F, F\rangle_{\mathbb{P}}.$$

Proposition 1. *For $f \in \mathbf{d}_m$ the random variable $I_m f$ $(= I_m(\tilde{f}))$ belongs to $\mathbb{D}$. The vector spaces D_m generated by $I_m f$ for $f \in \mathbf{d}_m$ are closed and orthogonal in $\mathbb{D}$. The sum*

$$\mathbb{D} = \mathbb{R} \bigoplus_{n \geqslant 1} D_n$$

is direct in the sense of the Hilbert structure of $\mathbb{D}$ $\quad (\|\cdot\|^2_{\mathbb{D}} = \|\cdot\|^2_{L^2} + \mathcal{E}[\cdot])$.

Any function F in $\mathbb{D}$ decomposes uniquely

$$F = \mathbb{E}[F] + \sum_{n \geqslant 1} I_n(f_n) \qquad \textit{with} \qquad f_n \in \mathbf{d}_n.$$

Proof. Let be $f = f_1(x_1) \cdots f_m(x_m) \in \mathbf{d}_m$ and $g = g_1(x_1) \cdots g_n(x_n) \in \mathbf{d}_n$. From $P_t I_m f = I_m p_t^{\otimes m} f$ comes

$$\begin{aligned}\mathcal{E}_t[I_m f, I_n g] &= \tfrac{1}{t}\langle I_m f - P_t I_m f, I_n g\rangle_{L^2(\mathbb{P})} = \tfrac{1}{t}\langle I_m(f - p_t^{\otimes m} f), I_n g\rangle \\ &= \delta_{mn} m! \langle \tfrac{f - p_t^{\otimes m} f}{t}, g\rangle_{L^2(\nu^{\times m})}.\end{aligned}$$

$F \in \mathbb{D}$ iff $\lim_{t\downarrow 0} \uparrow \mathcal{E}_t[F] < +\infty$ and $\mathcal{E}[F] = \lim_{t\downarrow 0} \mathcal{E}_t[F]$. Taking $f = g$, we get $I_m f \in \mathbb{D}$ and $\mathcal{E}[I_m f] = m! e_m[f]$. Then by density $I_n(f) \in \mathbb{D}$ $\forall f \in \mathbf{d_{n,sym}}$ and the orthogonality of the D_n follows.

The proof of the density of the Dirichlet chaos D_n is similar. □

For $F = \mathbb{E}[F] + \sum_{n \geqslant 1} I_n(f_n)$ in $\mathbb{D}$ this proof yields

$$\mathcal{E}[F] = \sum_{n \geqslant 1} n! e_n[f_n]. \tag{10}$$

II.3. A Mehler-type formula

The question is the following : is P_t positivity preserving ?

The idea is to attempt to write down rigorously the intuition that P_t describes the Markovian evolution of a family of independent particles which follow the bottom dynamics. We need some notation

The Poisson measure $N \odot \rho$: It is the Poisson measure N marked by independent marks with values in $(R, \mathcal{R})$ and with common law ρ. It is defined under $\mathbb{P} \times \hat{\mathbb{P}}$ where $\hat{\mathbb{P}} = \rho^{\mathbb{N}}$. It satisfies

$$\hat{\mathbb{E}} \exp \int \log G \, dN \odot \rho = \exp \int (\log \int G \, d\rho) \, dN \tag{11}$$

The Mehler-type formula

Thanks to measure theoretic lemmas, there exists a *Monte Carlo representation of* p_t : we can find $\zeta_t : X \times R \mapsto X$ measurable s.t. $\forall f \in \mathcal{L}^2(\nu)$

$$\int f(\zeta_t(x,r))\rho(dr) = (p_t f)(x) \quad \text{for } \nu\text{-a.e. } x$$

Then the Markovian evolution of the family of independent p_t-particles writes (with notation $\omega = \int \varepsilon_x N(dx)$)

$$Q_t F = \hat{\mathbb{E}} F \left(\int \varepsilon_{\zeta_t(x,r)} N \odot \rho(dxdr) \right). \tag{12}$$

Applying this to $F = \exp N \log(1+u)$ pour $-\frac{1}{2} \leqslant u \leqslant 0$ using (11) gives

$$Q_t F = \exp N \log(\int (1 + u(\zeta_t(x,r))\rho(dr)) = \exp N \log(1 + p_t u)$$

what shows by density that $Q_t = P_t$.

Theorem 2. *Suppose* $(X, \mathcal{X})$ *be s.t.* $\mathcal{X}$ *be separable and the atoms of* $\mathcal{X}$ *be the points of* X.

The semigroup P_t *is Markov,* $\mathcal{E}$ *is a Dirichlet form, the following Mehler formula holds : for* F *measurable non negative or bounded*

$$P_t F = \hat{\mathbb{E}} F\left(\int \varepsilon_{\zeta_t(x,r)} N \odot \rho(dxdr)\right). \tag{13}$$

The first lemma allows to pass from the pseudo-kernel p_t to a true kernel :

Lemma 3. *Suppose* $(X, \mathcal{X})$ *be such that* $\mathcal{X}$ *be separable and that the atoms of* $\mathcal{X}$ *be the points of* X. *There exist a true kernel* $\overline{p_t}$ *such that* $p_t f = \overline{p_t} f \quad \nu$*-a.e.*

Consequence of Dellacherie-Meyer [17] Chap I §11 et [18] Chap V §67. The second lemma allows to *simulate* the kernel $\overline{p_t}$:

Lemma 4. *Under the same hypothesis, there exists a measurable map* ζ_t *from* $(X \times [0,1], \mathcal{X} \times \mathcal{B}([0,1]))$ *into* $(X, \mathcal{X})$ *such that for all* f $\mathcal{X}$*-measurable non negative*

$$(\overline{p_t} f)(x) = \int f(\zeta_t(x,y))dy.$$

It is an application of El Karoui-Lepeltier [21] Thm 6 p123, noting that we can escape from Lusinian hypotheses thanks to a use of the (beautiful) Marczewski function cf. Thm 11 Chap I of Dellacherie-Meyer [17].

II.4. Gradients and carré du champ

In the same manner as N the Poisson random measure $N \odot \rho$ possesses a chaos representation. Let be $\hat{\mathcal{A}}$ the σ-field on $\Omega \times \hat{\Omega}$ generated by $N \odot \rho$. Then $\forall G \in L^2(\Omega \times \hat{\Omega}, \hat{\mathcal{A}}, \mathbb{P} \times \hat{\mathbb{P}})$

$$G = \mathbb{E}\hat{\mathbb{E}}G + \sum_{n \geqslant 1} J_n(g_n)$$

with $g_n \in L^2((\nu \times \rho)^n)$.

For $F \in \mathbb{D}$ with expansion $F = \sum_{n \geqslant 0} I_n(f_n)$, we define

$$F^\sharp = \sum_{n \geqslant 1} J_n((f_n)^{\flat_n}) \tag{14}$$

This is justified by (4) and proposition 1 applied to the J_n

$$\begin{aligned}\mathbb{E}\hat{\mathbb{E}}[(F^\sharp)^2] &= \sum n! \| f_n^{\flat_n} \|^2_{L^2((\nu \times \rho)^n)} \\ &= \sum n! \| \gamma_n[f_n] \|_{L^1((\nu)^n)} = 2\mathcal{E}[F].\end{aligned}$$

For $u \in \mathbf{d}$ using the expression of $(.)^{\flat_n}$ given above (cf (7))

$$(e^{N(\log(1+u))-\nu(u)})^\sharp = \sum_{n=1}^{+\infty} \frac{1}{n!} J_n((u^{\otimes n})^{\flat_n})$$

$$= e^{N(\log(1+u))-\nu(u)} \cdot \int u^\flat d\widetilde{N \odot \rho} = \sum_{n=1}^{+\infty} \frac{n}{n!} I_{n-1}((u^{\otimes(n-1)}) \int u^\flat d\widetilde{N \odot \rho}$$

so that thanks to the general formula

$$\hat{\mathbb{E}} \int F\, dN \odot \rho = \int (\int F\, d\rho)\, dN \tag{15}$$

we obtain what we needed to apply the preceding method with Friedrichs argument cf [15]

$$\begin{aligned}\hat{\mathbb{E}}[F^\sharp \overline{G^\sharp}] &= \textstyle\sum_{p,q} \lambda_p \overline{\mu_q} e^{i\tilde{N}(f_p-g_q)} \int (if_p)^\flat \overline{(ig_q)^\flat} dN d\rho \\ &= \textstyle\sum_{p,q} \lambda_p \overline{\mu_q} e^{i\tilde{N}(f_p-g_q)} N(\gamma(f_p, g_q))\end{aligned}$$

with $F = \sum_p \lambda_p e^{i\tilde{N}(f_p)}$, $G = \sum_q \mu_q e^{i\tilde{N}(g_q)}$ and the f_p and the g_q in $\mathbf{d}$.

Theorem 5. *The Dirichlet form $\mathcal{E}$ admits the bilinear operator $\Gamma[F,G] = \hat{\mathbb{E}}[F^\sharp G^\sharp]$ defined on $\mathbb{D} \times \mathbb{D}$ with values in $L^1(\mathbb{P})$ as OCC, it is a local form and the linear operator $(.)^\sharp$ defined on $\mathbb{D}$ with values in $L^2(\mathbb{P}\hat{\mathbb{P}})$ is a gradient for Γ.*

$$L^2(\mathbb{P}\hat{\mathbb{P}}, \hat{\mathcal{A}}) = \mathbb{R} + C_1(\mathbb{P}\hat{\mathbb{P}}) + \cdots + C_n(\mathbb{P}\hat{\mathbb{P}}) + \cdots \qquad F^{\sharp} = \textstyle\sum_n J_n(f_n^{\flat_n})$$

$$\Big\uparrow \, \sharp$$

$$\mathbb{D} = \mathbb{R} + \quad D_1 \quad + \cdots + \quad D_n \quad + \cdots \qquad F = \textstyle\sum_n I_n(f_n)$$

$$\cap \qquad\qquad \cap \qquad\qquad\qquad \cap$$

$$L^2(\mathbb{P}, \mathcal{A}) = \mathbb{R} + C_1(\mathbb{P}) + \cdots + C_n(\mathbb{P}) + \cdots \qquad F = \textstyle\sum_n I_n(f_n)$$

III. The lent particle formula and similar relations

III.1. The operators ε^+ and ε^-

$$\forall x, w \in \Omega,\ \varepsilon_x^+(w) = w\mathbf{1}_{\{x \in supp\, w\}} + (w + \varepsilon_x)\mathbf{1}_{\{x \notin supp\, w\}}$$
$$\forall x, w \in \Omega,\ \varepsilon_x^-(w) = w\mathbf{1}_{\{x \notin supp\, w\}} + (w - \varepsilon_x)\mathbf{1}_{\{x \in supp\, w\}}.$$

These operators extend to functionals by setting:

$$\varepsilon^+ H(w,x) = H(\varepsilon_x^+ w, x) \quad \text{and} \quad \varepsilon^- H(w,x) = H(\varepsilon_x^- w, x).$$

The image of $\mathbb{P} \times \nu$ by ε^+ is nothing but $\mathbb{P}_N$ whose image by ε^- is $\mathbb{P} \times \nu$:

Lemma 6. *Let H be $\mathcal{A} \otimes \mathcal{X}$-measurable and non negative, then*

$$\mathbb{E}\int \varepsilon^+ H d\nu = \mathbb{E}\int H dN \quad \textit{and} \quad \mathbb{E}\int \varepsilon^- H dN = \mathbb{E}\int H d\nu.$$

The measures $\mathbb{P} \times \nu$ and $\mathbb{P}_N = \mathbb{P}(d\omega)N(\omega)(dx)$ defined on the same space $(\Omega \times X, \mathcal{A} \times \mathcal{X})$ **are mutually singular**. The calculus with the lpf is very fast but one has to be carefull with negligible sets.

III.2. The LPF

$$\forall F \in \mathbb{D} \qquad F^\sharp = \int \varepsilon^-(\varepsilon^+ F)^\flat dN \odot \rho \tag{16}$$

This formula decomposes as

$$F \in \mathbb{D} \quad \overset{\varepsilon^+ - I}{\longmapsto} \quad \varepsilon^+ F - F \in \underline{\mathbb{D}} \quad \overset{\varepsilon^-((.)^\flat)}{\longmapsto} \quad \varepsilon^-((\varepsilon^+ F)^\flat) \in L^2_0(\mathbb{P}_N \times \rho) \quad \overset{d(N\odot\rho)}{\longmapsto} \quad F^\sharp \in L^2(\mathbb{P}\times\hat{\mathbb{P}})$$

where every operator is continuous on the range of the preceding one and where $L^2_0(\mathbb{P}_N \times \rho)$ is the closed set of elements G in $L^2(\mathbb{P}_N \times \rho)$ such that $\int_R G d\rho = 0$ $\mathbb{P}_N$-a.e.
And $F \in \mathbb{D}$

$$\Gamma[F] = \hat{\mathbb{E}}(F^\sharp)^2 = \int_X \varepsilon^-(\gamma[\varepsilon^+ F])\, dN.$$

Actually the lpf extends to $\mathbb{D}_{loc}$.

Applications : cf Bouleau-Denis JFA 2009; Hong-Kong 2009; PTRF 2010; POTA 2010 (submitted).

III.3. The operator $\delta_\sharp$ and the LPF for $\delta_\sharp$

Let us define $dom\,\delta_\sharp$ directly like this :

$$dom\,\delta_\sharp = \left\{ \begin{array}{l} G \in L^2(\hat{\mathcal{A}}, \mathbb{P}\hat{\mathbb{P}}) \text{ with expansion } G = \sum_{n \geqslant 0} J_n(g_n) \\ \text{ s.t. } g_n \in dom\,\delta_{\flat_n} \text{ and s.t.} \\ \sum_{n \geqslant 1} \|I_n(\delta_{\flat_n}(g_n))\|^2 = \sum n! \|\delta_{\flat_n}(g_n)\|^2_{L^2(\nu^n)} < +\infty \end{array} \right. \tag{17}$$

For $G \in dom\delta_\sharp$ we put

$$\delta_\sharp G = \sum_{n \geqslant 1} I_n(\delta_{\flat_n}(g_n))$$

Proposition 7. *The dual of the operator $(.)^\sharp$ with domain $\mathbb{D}$ is the operator $\delta_\sharp$ with domain $dom\,\delta_\sharp$ defined above.*

Proposition 8. *Let be $G = G(\omega, \hat{\omega}) \in dom\delta_\sharp$, the lent particle formula for $\delta_\sharp$ writes*

$$\delta_\sharp G = \int \varepsilon^-(\hat{\mathbb{E}}[\delta_\flat \varepsilon^+ G])\, d\tilde{N}$$

where the operator ε^+ deals with the random measure $N \odot \rho$ hence adds a point (x, r) while the operator ε^- deals with N and is the same as before.

In this formula N may be replaced by $\tilde{N}$ indifferently.

III.4. The LPF for the generator A

The Domain $\mathcal{D}A$ is characterized on the chaos by saying for $F=\sum I_n f_n$ that $f_n\in\mathcal{D}a_n$ and $\sum n!\|a_n(f_n)\|^2_{L^2}<\infty$. Then it is general that if $\mathcal{D}A$ is equipped by the norm $\|(I-A)F\|_{L^2}$ one has

$$\mathcal{D}A \overset{\sharp}{\longmapsto} dom\,\delta_\sharp \overset{\delta_\sharp}{\longmapsto} L^2$$

and $\delta_\sharp\sharp$ is continuous and equal to $-2A$.

Applying the LPF to $\sharp$ and then to $\delta_\sharp$ and dealing carefully with ε^+ et ε^- one gets

$$\forall F\in\mathcal{D}A \qquad\qquad \varepsilon^+F\in\mathcal{D}a \quad \text{et} \quad AF=\int\varepsilon^-a\varepsilon^+F\,d\tilde{N} \qquad (18)$$

III.6. Historical origin of the LPF

This formula appeared in works on quantum physics around creation and annihilation operators : about bosons by Fichtner and Freudenberg (1987) [22] and formula 4.19 of [23] where it expresses the second quantization operation denoted $d\mathbf{\Gamma}$ (operation which associates $\sharp$ to $\flat$ or also A to a):

$$(d\mathbf{\Gamma}K)F=\int \tilde{N}\varepsilon^{-}K(\varepsilon^{+}-I)F \tag{19}$$

Such formulas have been encountered or suggested by many authors with various notation Albeverio, Kondratiev, Röckner [2], Privault [43] (formula before prop 8), also Hitsuda [27], Kabanov [30], Nualart, Vivès [37], Picard [39] and [40], etc.

Our contribution is only in the proof of the validity of the LPF for $\sharp$ and $\flat$ on the whole domain $\mathbb{D}$ of the form and the observation that it allows concrete computations provided that we are careful with respect to singular measures generated by the operators ε^{+} and ε^{-}.

III.7. Recipes

- Si $g = g(x,r)$ bounded with small norm and in the domain of $\delta_\flat$

$$\delta_\sharp e^{N\odot\rho\log(1+g)} = e^{N\log\int(1+g)d\rho} \cdot N\left(\frac{\delta_\flat g}{\int(1+g)d\rho}\right) \tag{20}$$

- Si $f = f(x)$, $g = g(x,r)$

$$\delta_\sharp[e^{N(f)}N\odot\rho(g)] = e^{N(f)}\int e^{-f}\delta_\flat(e^f g)dN \tag{21}$$

Lemma 10. *Let be $u \in dom\delta_\flat$ bounded with small norm*

$$\delta_\sharp e^{N\odot\rho(\log(1+u))} = \delta_\sharp\left[e^{N[\log(1+\int ud\rho)]}N\odot\rho[\frac{u}{1+\int ud\rho}]\right]$$

Corollary 11. *If $G(\omega,\hat{\omega}) \in dom\delta_\sharp$, there exists $K(\omega,x,r)$ such that $\int Kd\rho = 0$ and s.t.*

$$\delta_\sharp G = \delta_\sharp\left[\int K(\omega,x,r)dN\odot\rho(x,r)\right] = \delta_\sharp\left[\int K(\omega,x,r)d\widetilde{N\odot\rho}(x,r)\right]$$

- More hidden formulas

  Let be $f = f(x,r) \in L^1(\nu \times \rho)$, how to compute $\mathbb{E}[N \odot \rho(f)]$?
  Clearly the series $\sum_i \nu(f(.,r_i))$ diverges since the marks r_i are i.i.d.

Lemma 12. *Let be $f = f(x,r) \geqslant 0$ then in the sense of equality in distribution*

$$\mathbb{E}e^{-N\odot\rho(f)} = \lim_{A\uparrow X} \downarrow e^{-\nu(A)} \left(1 + \int_A e^{-f(.,r_1)} d\nu \left(1 + \frac{1}{2}\int_A e^{-f(.,r_2)} d\nu \left(1 + \frac{1}{3}\int_A e^{-f(.,r_3)} d\nu \,(\ldots)\right)\right)\right)$$

$$\mathbb{E}[N \odot \rho(f)] = \lim_{t\uparrow\infty} \uparrow e^{-t} \Big[\int f(.,r_1) d\nu \Big(\frac{1}{1!} + \frac{t}{2!} + \frac{t^2}{3!} + \cdots\Big) + \\ + \int f(.,r_2) d\nu \Big(\frac{t}{2!} + \frac{t^2}{3!} + \cdots\Big) + \int f(.,r_3) d\nu \Big(\frac{t^2}{3!} + \cdots\Big) + \cdots\Big]$$

The fact that the equality cannot be better than in law is seen by the fact that the left hand side doesn't involve the order of the r_i's but the right hand side does.

IV. Equivalences of norms

IV.1. Subordination and Fock space

Let μ_t be the law of a subordinator of Bernstein function φ.

$$
\begin{array}{ccccc}
\mathcal{E} & & \mathcal{E}_2 & & \mathcal{E}_1 \\
\uparrow & & \uparrow & & \uparrow \\
P_t & \rightsquigarrow & Q_t = \int P_\alpha \mu_t(d\alpha) & & \Pi_t \\
\uparrow & & & & \uparrow \\
p_t & & \rightsquigarrow & & \pi_t = \int p_\alpha \mu_t(d\alpha)
\end{array}
$$

To $Q_t = \int P_\alpha \mu_t(d\alpha)$ corresponds a Dirichlet form $\mathcal{E}_2 \neq \mathcal{E}_1$.

With the notation of second quantization denoting $\mathbf{\Gamma} K$ the operator obtained by lifting the operator K as (p_t) is lifted to P_t, and $d\mathbf{\Gamma} K$ the operator obtained by lifting the operator K as a is lifted to A:

$$-\varphi(-d\mathbf{\Gamma} a) \neq d\mathbf{\Gamma}(-\varphi(-a)).$$

Here in fact $\mathcal{E}_1 \leqslant \mathcal{E}_2 \leqslant \varphi'(0)\mathcal{E}$. (due to the fact that Bernstein functions are concave and subadditive)

We see that an equivalence of norms $\|.\|_{L^p(\mathbb{P})}$ between $-\sqrt{-A}(F)$ and $\sqrt{\Gamma[F]}$ cannot be directly obtained from a similar equivalence between the norms $\|.\|_{L^p(\nu)}$ de $-\sqrt{-a}(F)$ et $\sqrt{\gamma[F]}$.

IV.2. Factorial measures

The random measure $N{\times}N$, which isn't Poisson, has as intensity measure $\nu\times\nu+\nu|_\Delta$ where $\nu|_\Delta$ is the image measure of ν by $x\mapsto(x,x)$.

We note measures acting on functions as usual in the theory of Markov processes.

Lemma 13. *The random measure on $(X,\mathcal{X})^2$ given by*

$$f\mapsto\int N(dy)\varepsilon_y^- N(dx)f(x,y)=\int N(dy)\varepsilon_y^- N(dx)\varepsilon_x^- f(x,y)$$

has for intensity measure $\nu\times\nu$.

The random measure $f\mapsto\int(N\varepsilon^-)^k f$ on $(X,\mathcal{X})^k$ has for intensity measure ν^k.

Lemma 6 extends like this

Lemma 14. a) *Let $H(\omega,x,y)$ be $\mathcal{A}\times\mathcal{X}\times\mathcal{X}$-measurable non-negative.*

$$\mathbb{E}\int N(dy)\varepsilon_y^- N(dx)\varepsilon_x^- H=\mathbb{E}\int H d\nu d\nu$$

If H is $\mathcal{A}\times(\mathcal{X})^k$-measurable non-negative

$$\mathbb{E}\int(N\varepsilon^-)^k H=\mathbb{E}\int H d\nu^{\times k} \tag{22}$$

Factorial measures dissipate some mysteries of multiple integrals:

Proposition 15. *Let f bounded and in $L^1(\nu^{\times n})$*

$$(\tilde{N}\varepsilon^-)^n f = I_n(f) \tag{23}$$

and relation (23) *extends to $f \in L^2(\nu^{\times n})$.*

It follows of course that

$$\|(\tilde{N}\varepsilon^-)^n f\|_2^2 = n!\|\tilde{f}\|_2^2 \tag{24}$$

Relations (23) et (24) express I_n as true multiple integrals without using the simplex $s_1 < s_2 < \cdots < s_k$. The first formulas simpler on the Poisson space than on the Wiener space !
Remark. It is possible to *define* the multiple integrals starting from the $(\tilde{N}\varepsilon^-)^n$ by using the relation

$$\varepsilon^+(\tilde{N}\varepsilon^-)^n u^{\otimes n} = (\tilde{N}\varepsilon^-)^n u^{\otimes n} + nu(\tilde{N}\varepsilon^-)^{n-1} u^{\otimes(n-1)} \tag{25}$$

IV.3. Inequalities

When we write $F^{\sharp\sharp}$ the second $\sharp$-operator acts on $F^{\sharp}(\omega, \hat{\omega}_1)$ with fixed $\hat{\omega}_1$ and adds a new $\hat{\omega}_2$ independently. We write $\hat{\mathbb{E}}$ for the expectation with respect to all these $\hat{\omega}_1$, $\hat{\omega}_2$ etc., in other words $\hat{\mathbb{E}}$ denotes the expectation with respect to $\hat{\mathbb{P}}^{\otimes \mathbb{N}^*}$.

Now, we introduce the following notation for any $F \in \mathbb{D}^{k,2}$:

$$\Gamma_k[F] = \hat{\mathbb{E}}[(F^{(k\sharp)})^2] \tag{26}$$

This is a general definition of the carré du champ operator of order k (cf. P.-A. Meyer sém XVIII [34] p182 in the Ornstein-Uhlenbeck case where operators Γ_k satisfy a specific recurrence relation due to a commutation identity, that we do not suppose in the present subsection).

Proposition 16. *We can choose $(R, \mathcal{R}, \rho)$ and the gradient operator $\flat$ such that for all $k \in \mathbb{N}^*$, $p \geqslant 1$ the following inequality holds for any $F \in \mathbb{D}^{k,p}$:*

$$A_p \|(\Gamma_k[F])^{1/2}\|_{L^p(\mathbb{P})} \leqslant \|F^{k\sharp}\|_{L^p(\mathbb{P}\times\hat{\mathbb{P}}^k)} \leqslant B_p \|(\Gamma_k[F])^{1/2}\|_{L^p(\mathbb{P})}. \tag{27}$$

where A_p and B_p are the constants appearing in the Khintchine inequality.

The preceding argument gives also the following result which makes a connection with the "martingale transforms" method cf. Banuelos [6]:

Proposition 17. *Let us suppose that ν be a Lévy measure on $(X, \mathcal{X}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and that ρ be the Lebesgue measure on $[0,1]$.*

For $1 \leqslant p < \infty$, the centered Lévy processes

$$Y_t = \int 1_{[0,t]}(s) y \tilde{N}(ds, dy)$$

,

$$Z_t^1 = \int 1_{[0,t]}(s) y (2\theta - 1) \widetilde{N \odot \rho}(ds, dy, d\theta)$$

$$Z_t^2 = \int 1_{[0,t]}(s) y \psi(\theta) \widetilde{N \odot \rho}(ds, dy, d\theta)$$

where ψ is the inverse of the distribution function of the standard Gaussian measure, and

$$Z_t^3 = \int 1_{[0,t]}(s) y e^{2i\pi\theta} \widetilde{N \odot \rho}(ds, dy, d\theta)$$

are martingales with equivalent $\mathcal{H}^p$-norms (quadratic or maximal).

Proof. By Burkholder-Davis-Gundy inequalities ([18] p.304) the quadratic and maximal $\mathcal{H}^p$-norms are equivalent. Let us note first that $[Z^1, Z^1]_t \leqslant [Y, Y]_t$, hence the $\mathcal{H}^p$-norms of Z^1 are bounded by those of Y.

Then, let $2\theta - 1 = \sum_j \frac{\xi_j}{2^j}$ be the expansion of $2\theta - 1$ on the Rademacher functions of $([0,1], \rho)$ and $U_t^\varepsilon = \int 1_{[0,t]}(s) 1_{\{|y|>\varepsilon\}} y(2\theta - 1)\ N \odot \rho(ds, dy, d\theta)$.

Khintchine inequality gives (thanks to the fact that $\sum_j (\frac{1}{2^j})^2 = \frac{1}{3}$)

$$\frac{A_p}{\sqrt{3}} \|(\int 1_{[0,t]}(s) 1_{\{|y|>\varepsilon\}} y^2\ N(ds, dy))^{1/2}\|_p \leqslant \|U_t^\varepsilon\|_p \leqslant \frac{B_p}{\sqrt{3}} \|(\int 1_{[0,t]}(s) 1_{\{|y|>\varepsilon\}} y^2\ N(ds, dy))^{1/2}\|_p$$

and by $U_t^\varepsilon = \int 1_{[0,t]}(s) 1_{\{|y|>\varepsilon\}} y(2\theta - 1)\ \widetilde{N \odot \rho}(ds, dy, d\theta)$ since $\int (2\theta - 1) d\rho = 0$, this gives (letting ε go to zero, extracting a subsequence and applying the dominated convergence theorem)

$$\frac{A_p}{\sqrt{3}} \|[Y, Y]_t^{1/2}\|_{L^p} \leqslant \|Z_t^1\|_{L^p} \leqslant \frac{B_p}{\sqrt{3}} \|[Y, Y]_t^{1/2}\|_{L^p}.$$

and the left hand side give the majorization of the quadratic $\mathcal{H}^p$-norm of Y in terms of the of the maximal $\mathcal{H}^p$-norm of Z^1.

To deal with Z^2, let us remark that $\psi(\theta)$ is Gaussian under $\rho(d\theta)$. It follows that Z^2_t is conditionally Gaussian given ω, and $(\int |Z^2_t|^p \, d\hat{\mathbb{P}})^{1/p} = c_p[Y,Y]_t^{1/2}$ what gives the result.

Lastly, for Z^3 we have to use an interesting extension of Khintchine inequality, cf Peškir [38], Finch [24], Latala [31] : let a_j be complex numbers, and θ_i be i.i.d. uniformly distributed on $[0,1]$ (Steinhaus sequence) then there are constants $0 < c_p,\ C_p < \infty$ such that

$$c_p(\sum |a_j|^2)^{1/2} \leqslant \| \sum a_j e^{2i\pi\theta_j} \|_{L^p} \leqslant C_p(\sum |a_j|^2)^{1/2} \tag{28}$$

and the argument follows the case of Z^1. □

Non completely understood questions in progress

• In the case $\mathbb{R}_+ \times X$, the connexion between the previsible representation and the lpf $F^\sharp = \int (H^\sharp + H^\flat)\widetilde{dN \odot \rho} = \int \varepsilon^-(\varepsilon^+ F)^\flat \widetilde{dN \odot \rho}$.

• What about operators lending two particles or infinitely many particles...

• By the Fock space there is a one to one map between $L^2_{Poisson}$ and L^2_{Wiener}. In the simple case $X = [0,1]$ and $\nu = \rho = dx$ with $H^1([0,1])$ and $\gamma[u] = u'^2$, we get an "extended Mehler type" structure on the Wiener space and a one to one map betwen $\mathbb{D}_{Poisson}$ and $\mathbb{D}_{Wiener}$. Also the gradients may be transported...

• The operator $\delta_\sharp$ thought as a stochastic integral acts on functionals of Lévy processes in $\mathbb{R}^{d+1}$ and gives functionals of Lévy processes in $\mathbb{R}^d$.

• In the case where the bottom space is $\mathbb{R}^d$ with Lebesgue measure and usual gradient ($\mathbb{D} = H^1(\mathbb{R}^d)$. Schwartz distributions down can be lifted into Watanabe distributions above (or Meyer-Yan distributions) ... Ostrogradski formula $\int_A divG = \int_S n \cdot G$ becomes $\nu[1_A \delta G] = \nu\rho[(1_A)^\flat G]$

• By the fact that $(R, \mathcal{R}, \rho)$ (the mark space) is a probability space, the countable iteration $(\sharp)^\infty$ could have sense...

• The main features of the method can be extended to non-Poisson cases by considering marked random point measures. (e.g. isotropic processes cf Bouleau *Sém. Proba XLIII*, 2010)

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