Malliavin Greeks for complex Asian options in a jump diffusion setting

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From a paper with Valerio Marchisio

1. The problem

Ingredients

- a Brownian motion W on $\mathbb{R}$;
- a Poisson process J with intensity $\lambda > 0$:

$$J_t = \sum_{n\geq 1} \mathbf{1}_{\{T_n \leq t\}}$$

where $T_1, T_2 - T_1, T_3 - T_2, \dots$ are i.i.d. of law Exp(λ);

- a sequence $\Delta_1, \Delta_2, \dots$ of i.i.d. r.v.'s such that $\Delta_i \in L^p$ $\forall p$ and the law of Δ_i is a.c. w.r.t. the Lebesgue measure - we set g as the common density.

We assume that

$$W, \{T_n\}_n, \{\Delta_n\}_n \text{ are independent.}$$

$\{\Delta_n\}_n$ and $\{T_n\}_n$ give the jump amplitudes and the jump times respectively of the compound Poisson process

$$N_t = \sum_{n=1}^{J_t} \Delta_n.$$

1. The problem

Fix $b = b(t,x)$, $\sigma = \sigma(t,x)$ and $c = c(t,a,x)$ "good enough". Set

$$X_t = x + \int_0^t b(s,X_s)ds + \int_0^t \sigma(s,X_s)dW_s + \sum_{n=1}^{J_t} c(T_n, \Delta_n, X_{T_n^-})$$

and

$$Y_t = \int_0^t X_s ds.$$

GOAL: find representation formulas for

$$\partial_x \mathbb{E}(f(X_t, Y_t))$$

in terms of expectations involving $f(X_t, Y_t)$.

$\rightsquigarrow$ Link to the delta of European options of COMPLEX Asian type, i.e. written on both the underlying asset price X_t and its integral mean Y_t on $[0,t]$.

As $n \geq 1$ we get

$$\partial_x \mathbb{E}\big(f(X_t, Y_t)\mathbf{1}_{\{J_t=n\}}\big) = \mathbb{E}\big(f(X_t, Y_t)\Theta\mathbf{1}_{\{J_t=n\}}\big) + \mathbb{E}\big([f(X_t, Y_t), W]_\pi \mathbf{1}_{\{J_t=n\}}\big) \qquad (1)$$

where:

- Θ is a "standard" Malliavin weight

  $\rightsquigarrow$ Skorohod integral of a suitable process: $\Theta = \delta(W)$;

- $[\cdot, \cdot]_\pi$ is a border term operator

  $\rightsquigarrow$ arising when dealing with $\{\Delta_n\}_n$ and $\{T_n\}_n$, i.e. the jump noise;

- $\pi = (\pi_1, \ldots, \pi_n)$ is a set of weights in a suitable scalar product: $\langle U, V\rangle_\pi = \sum_{i=1}^n U_i V_i \pi_i$, $U, V \in \mathbb{R}^n$.

  $\rightsquigarrow$ again coming from the jump noise.

Formula (1) follows from a suitable composition of IBP formulas from standard Gaussian Malliavin calculus (from the Brownian noise) with the Malliavin calculus developed in [BalBavM] w.r.t. locally smooth laws, i.e. the ones from the jump noise $\{\Delta_n\}_n$ and $\{T_n\}_n$.

Remark

1. No weight is available from the composition of the Brownian Malliavin calculus and the Malliavin calculus w.r.t. the jump times: when $\sigma \neq 0$, in general X_t is not derivable w.r.t. the jump times.

2. The case $f(x,y) \equiv f(y)$ has been widely studied but the general case is not - there are few results, available in Benhamou [LSE preprint, 2000] for pure diffusions.

The complication is not trivial: (X_t, Y_t) is a degenerate jump-diffusion process

$\rightsquigarrow$ a standard use of the Malliavin calculus gives weights that are unfeasible from the numerical point of view. Here, we aim to find a formula of the type (1) that one can really implement in practice.

3. In the literature, an effort is done to take π such that $[\cdot,\cdot]_\pi = 0$. Here, we choose $\pi_i = 1\ \forall i$: the border term operator affects the formula and this gives efficient numerical results.

MAIN STEPS

Malliavin derivative on A + Skorohod integral on A

$\Downarrow$

duality relationship on A + chain rule on A

$\Downarrow$

IBP formulas on A:

$$\mathbb{E}\big(\partial_k\Phi(F)G\mathbf{1}_A\big) = \mathbb{E}\big(\Phi(F)H_i(F;G)\mathbf{1}_A\big) + \mathbb{E}\big([\Phi(F),\Lambda_k(F,G)]_\pi\mathbf{1}_A\big)$$
$$\forall\Phi\in C_b^1(\mathbb{R}^d),\quad k=1,\ldots d.$$

Then,

$$\begin{aligned}\partial_\vartheta\mathbb{E}\big(\Phi(F)\mathbf{1}_A\big) &= \mathbb{E}\Big(\sum_{k=1}^d \partial_k\Phi(F)\partial_\vartheta F^k\mathbf{1}_A\Big)\\ \text{[IBP } \to\text{]} &= \mathbb{E}\Big(\Phi(F)\sum_{k=1}^d H_k(F,\partial_\vartheta F^k)\mathbf{1}_A\Big) + \mathbb{E}\Big(\Big[\Phi(F),\sum_{k=1}^d \Lambda_k(F,\partial_\vartheta F^k)\Big]_\pi\mathbf{1}_A\Big)\end{aligned}$$

2. Malliavin regularity

Set $\chi_s^{u,\chi}$ the solution to

$$\chi_s = \chi + \int_u^s b(r,\chi_r)ds + \int_u^s \sigma(r,\chi_r)dB_r.$$

On $\{J_t = n\}$,

$$X_s = \begin{cases} X_s = \chi_s^{0,x} + \mathbf{1}_{\{s=T_1\}}\, c(T_1,\Delta_1,\chi_{T_1}^{0,x}) & \text{if } s \in [0,T_1] \\ \chi_s^{T_{k-1},X_{T_{k-1}}} + \mathbf{1}_{\{s=T_k\}}\, c(T_k,\Delta_k,\chi_{T_k}^{T_{k-1},X_{T_{k-1}}}) & \text{if } s \in (T_{k-1},T_k] \text{ and } k = 2,\ldots,n \\ \chi_s^{T_n,X_{T_n}} & \text{if } s \in (T_n,t] \end{cases}$$

so that

$$X_t = f(\omega,\Delta_1,\ldots,\Delta_n,T_1,\ldots,T_n)$$
$$Y_t = h(\omega,\Delta_1,\ldots,\Delta_n,T_1,\ldots,T_n)$$

with $f,h : \Omega \times \mathbb{R}^n \times \Xi^n \to \mathbb{R}$ is $\sigma(W_s\,;\, s \le t) \times \mathscr{B}(\mathbb{R}^2) \times \mathscr{B}(\Xi^n)$-measurable, where $\Xi^n = \{u \in \mathbb{R}_+^n\,;\, u_1 < \cdots < u_n\}$.

For $a = (a_1,\ldots,a_n)$ and $u = (u_1,\ldots u_n)$, the regularity of $(a,u) \mapsto f(\omega,a,u), h(\omega,a,u)$ and the Malliavin regularity w.r.t. the Brownian noise of $\omega \mapsto f(\omega,a,u), h(\omega,a,u)$ both depend on suitable regularity conditions on the diffusion coefficients b,σ,c.

Assumption R. • R1 *$(r,x)\mapsto b(r,x),\sigma(r,x),c(r,a,x)$ are continuous and $x\mapsto \sigma(r,x),\mapsto c(r,a,x)$ are twice differentiable, with bounded derivatives of first and second order, and having linear growth with respect to x, uniformly with respect to t and a.*

• R2. *There exists $\eta>0$ such that, for any r,a,x,*

$$|1+\partial_x c(r,a,x)|\geq \eta.$$

Under R, $\xi=\partial_x X$ and $\widehat{\xi}=\xi^{-1}$ are both well defined and

$$\xi_t = 1+\int_0^t \partial_x b(r,X_r)\xi_r dr+\int_0^t \partial_x\sigma(r,X_r)\xi_r dW_r+\sum_{i=1}^{J_t}\partial_x c(T_i,\Delta_i,X_{T_i-})\xi_{T_i-}$$

$$\widehat{\xi}_t = 1-\int_0^t \widehat{\xi}_r\big(\partial_x b-(\partial_x\sigma)^2\big)(r,X_r)dr-\int_0^t\widehat{\xi}_r\partial_x\sigma(r,X_r)dW_r+ \\ -\sum_{i=1}^{J_t}\ \widehat{\xi}_{T_i-}\frac{\partial_x c(T_i,\Delta_i,X_{T_i-})}{1+\partial_x c(T_i,\Delta_i,X_{T_i-})}.$$

If $\beta_t=\partial_x Y_t=\int_0^t\xi_s ds$, Assumption R implies that $X_t,Y_t,\xi_t,\beta_t\in L^p$ for all p.

1. Derivative w.r.t. the Brownian noise

We set $\mathbb{D}_0^{1,p}(\{J_t = n\})$ as the space of the r.v.'s of the form

$$F = f(\omega, \Delta_1, \Delta_n, T_1, \dots, T_n) \quad \text{for} \quad \omega \in \{J_t = n\}$$

such that the functional $\omega \mapsto f(\omega, a, u)$ belongs to $\mathbb{D}^{1,p}$ and setting

$$D_0 F = Df(\cdot, a, u)\big|_{a = (\Delta_1, \dots, \Delta_n), u = (T_1, \dots, T_n)}$$

one has $F\mathbf{1}_{\{J_t=n\}} \in L^p(\Omega)$ and $D_0 F\mathbf{1}_{\{J_t=n\}} \in L^p(H)$, being $H = L^2([0,t], \mathscr{B}([0,t]), \text{Leb})$.

Similarly, $\mathbb{D}_0^{k,p}(\{J_t = n\})$ is defined.

One has

Under Assumption R, $X_t, Y_t \in \mathbb{D}_0^{2,p}(\{J_t = n\})$ and $\xi_t, \beta_t \in \mathbb{D}_0^{1,p}(\{J_t = n\})$, $\forall p$. Moreover, on the set $\{J_t = n\}$, for $s \leq t$

$$D_{0,s} X_t = \xi_t \xi_s^{-1} \sigma(s, X_s), \quad D_{0,s} Y_t = \xi_s^{-1} \sigma(s, X_s) \int_s^t \xi_r dr.$$

2. Derivative w.r.t. the jump amplitudes

We set $\mathcal{S}^{\Delta}_{n,k}(\{J_t=n\})$ as the space of the r.v.'s of the form

$$F=f(\omega,\Delta_1,\dots,\Delta_n,T_1,\dots,T_n)\quad \text{for}\quad \omega\in\{J_t=n\},$$

such that the function $a\mapsto f(\omega,a,u)$ belongs to $C^k(\mathbb{R}^n)$.

For $F\in\mathcal{S}^{\Delta}_{n,k}$, set

$$D^{\Delta}_i F=\partial_{a_i}f(\cdot,a,u)\big|_{a=(\Delta_1,\dots,\Delta_n),\,u=(T_1,\dots,T_n)}$$

Assumption R(Δ). $a\mapsto c(r,a,x)$ is C^1 and there exists $\eta>0$ such that

$$|\partial_a c(r,a,x)|\geq\eta\quad\forall\ r,a,x.$$

One has

Under R + R(Δ), $X_t,Y_t\in\mathcal{S}^{\Delta}_{n,2}(\{J_t=n\})$ and $\xi_t,\beta_t\in\mathcal{S}^{\Delta}_{n,1}(\{J_t=n\})$. Moreover,

$$D^{\Delta}_i X_t=\mathbf{1}_{\{T_i<t\}}\,\xi_t\xi^{-1}_{T_i}\partial_a c(T_i,\Delta_i,X_{T_i^-}),\qquad D^{\Delta}_i Y_t=\mathbf{1}_{\{T_i<t\}}\,\xi^{-1}_{T_i}\partial_a c(T_i,\Delta_i,X_{T_i-})\int_{T_i}^t\xi_r dr.$$

3. Derivative w.r.t. the jump times

We set $\mathcal{S}^T_{n,k}(\{J_t = n\})$ as the space of the r.v.'s of the form

$$F = f(\omega, \Delta_1, \Delta_n, T_1, \dots, T_n) \quad \text{for} \quad \omega \in \{J_t = n\},$$

such that the function $u \mapsto f(\omega, a, u)$ belongs to $C^k(\Xi_n)$.

For $F \in \mathcal{S}^T_{n,k}$, set

$$D^T_i F = \partial_{u_i} f(\cdot, a, u)\big|_{a = (\Delta_1, \dots, \Delta_n), u = (T_1, \dots, T_n)}$$

Assumption R(T). $r \mapsto c(r, a, u)$ is C^1 and there exists $\eta > 0$ such that

$$|q(r, a, x)| \geq \eta \quad \forall \ r, a, x.$$

where

$$q(r, a, x) = \Big(\partial_r c + b(1 + \partial_x c)\Big)(r, a, x) - b(r, x + c(r, a, x)).$$

One has

If $\sigma = 0$, under R + R(T), $X_t, Y_t \in \mathcal{S}^T_{n,2}(\{J_t = n\})$ and $\xi_t, \beta_t \in \mathcal{S}^T_{n,1}(\{J_t = n\})$. Moreover,

$$D^T_i X_t = \mathbf{1}_{\{T_i < t\}}\, \xi_t \xi^{-1}_{T_i} q(T_i, \Delta_i, X_{T_i^-}), \qquad D^T_i Y_t = \mathbf{1}_{\{T_i < t\}}\, \xi^{-1}_{T_i} q(T_i, \Delta_i, X_{T_i-}) \int_{T_i}^t \xi_r dr.$$

3. Sensitivity formulas from the Brownian noise

We set $Dom_{n,t,p}(\delta_0)$ as the space of the processes of the form

$$U_s \mathbf{1}_{\{J_t=n\}} = U_s(\omega, \Delta_1, \ldots, \Delta_n, T_1, \ldots, T_n) \quad \text{for} \quad \omega \in \{J_t = n\}$$

such that $(\omega, s) \mapsto U_s(\omega, a, u)$ belongs to $Dom_p(\delta)$ and setting

$$\delta_0(U) = \delta\big(U(\cdot, a, u)\big)\Big|_{a = (\Delta_1, \ldots, \Delta_n), u = (T_1, \ldots, T_n)}$$

one has $U\mathbf{1}_{\{J_t=n\}} \in L^p(H)$ and $\delta_0(U)\mathbf{1}_{\{J_t=n\}} \in L^p(\Omega)$.

• Brownian Malliavin integration by parts formula

Let $F=(F^1,\dots,F^d)$ *be such that* $F^i\in\mathbb{D}_0^{2,p}$ *and let* $U^1,\dots U^d\in Dom_{n,t,p}(\delta_0)$ $\forall p$. *Let* $\sigma_{U,DF}$ *be the random matrix defined as*

$$\sigma_{U,D_0F}^{\ell k}=\langle U^\ell,D_0F^k\rangle_H=\int_0^t U_s^\ell D_{0,s}F^k\,ds,\qquad \ell,k=1,\dots,d$$

and suppose that

$$|\det\sigma_{U,D_0F}|^{-1}\mathbf{1}_{\{J_t=n\}}\in L^p(\Omega)\quad\forall p. \tag{2}$$

Set $\widehat{\sigma}_{U,D_0F}=\widehat{\sigma}_{U,D_0F}^{-1}$ *and let* $G\in\mathbb{D}_0^{1,p}$ $\forall p$*. Then, for any* $\Phi\in C_p^1(\mathbb{R}^d)$*,*

$$\mathbb{E}\big(\partial_\ell\Phi(F)G\mathbf{1}_{\{J_t=n\}}\big)=\mathbb{E}\Big(\Phi(F)\delta_0\Big(\sum_{k=1}^d\widehat{\sigma}_{U,DF}^{\ell k}U^kG\Big)\mathbf{1}_{\{J_t=n\}}\Big),\qquad \ell=1,\dots,d.$$

The case $U^k=DF^k$ gives $\sigma_{U,DF}=\sigma_F$, i.e. the Malliavin covariance matrix, and (2) is the classical non-degeneracy condition.

- Sensitivity w.r.t. x

Theorem 0 *Let Assumption R hold. Set* $Z_t=(X_t,Y_t)$. *Let* $U^1,U^2\in Dom_{n,t,p}(\delta_0)$ *and for* $\ell,k=1,2$, *let* $\sigma^{\ell k}_{U,D_0Z_t}=\langle U^\ell, D_0F^k\rangle_H$. *Suppose that*

$$|\det\sigma_{U,D_0Z_t}|^{-1}\mathbf{1}_{\{J_t=n\}}\in L^p(\Omega)\quad\forall p \tag{3}$$

and on the set $\{J_t=n\}$, *set* $\widehat{\sigma}_{U,D_0Z_t}=\sigma^{-1}_{U,D_0Z_t}$. *Then, for any* Φ *with polynomial growth one has*

$$\partial_x\mathbb{E}\big(\Phi(X_t,Y_t)\mathbf{1}_{\{J_t=n\}}\big)=\mathbb{E}\Big(\Phi(X_t,Y_t)\delta_0(W)\mathbf{1}_{\{J_t=n\}}\Big)$$

where

$$W=\sum_{k=1}^{2}\widehat{\sigma}^{1k}_{U,DZ_t}U^k\xi_t+\sum_{k=1}^{2}\widehat{\sigma}^{2k}_{U,DZ_t}U^k\beta_t.$$

- The formula in practice

The choice $U = D_0 Z_t$ (i.e. σ_{U,D_0Z_t}= Malliavin covariance matrix) is unfeasible from the practical point of view. Moreover, thinking e.g. to the pure diffusion case, the non degeneracy condition is not obvious, unless the Hörmander condition is satisfied.

Then, we look for U such that σ_{U,D_0Z_t} is "simple" (triangular).

Set

$$\mathscr{A}_t^W = \Big\{\alpha\,:\,[0,t]\to\mathbb{R} \text{ such that } \int_0^t \alpha_u du = 0\Big\}$$

and

$$U_r^1 = \xi_r\sigma^{-1}(X_r)\alpha_r, \qquad U_r^2 = \xi_r\sigma^{-1}(X_r)$$

This gives

$$\sigma_{U,DZ_t} = \begin{pmatrix} 0 & -\int_0^T \beta_u\alpha_u du \\ T\xi_T & T\beta_T - \int_0^T \beta_u du \end{pmatrix} \text{ and } W_s = \frac{\xi_s\sigma^{-1}(X_u)}{t}\Big(1 - \frac{\int_0^t \beta_r dr}{\int_0^t \beta_r\alpha_r dr}\,\alpha_s\Big).$$

One has

Proposition *If* $A_s = \int_0^s \alpha_r dr \geq 0$, *(3) holds.*

In practice, we take, for $s \in [0, t]$, either $\alpha_s = t/2 - s$ or $\alpha_s = \sin\left(2\pi \frac{s}{t}\right)$, for which $A_s = \int_0^s a_u \, du \geq 0$.

Remark We can compare with [Ben]: his weight is $\delta(W)$ with

$$W_u = \xi_u \sigma^{-1}(X_u) \left(\frac{\beta_T^2 + 2\xi_u (\int_0^T r\xi_r dr - T\beta_T)}{\beta_T (2 \int_0^T r\xi_r dr - T\beta_T)} \right).$$

One has that W can be found through

$$U_s^1 = \xi_s^2 \sigma^{-1}(X_s), \qquad U_s^2 = \xi_s \sigma^{-1}(X_s).$$

4. Sensitivity formulas from the jump noise

Fix $t > 0$ and $n \geq 1$.

The pair (X_t, Y_t) is measurable w.r.t.

$$\sigma(B_s\,;\ s \leq t) \vee \sigma(J_t) \vee \sigma(\Delta_1, \Delta_2, \ldots) \vee \sigma(T_1, T_2, \ldots)$$

and on the set $\{J_t = n\}$ one has

$$\Phi(X_t, Y_t) = \varphi(\omega, \Delta_1, \ldots, \Delta_n, T_1, \ldots, T_n).$$

The choices

1. $\{V_i\} = \{\Delta_i\}$ and $\mathscr{G} = \sigma(B_s\,;\ s \leq t) \vee \sigma(J_t) \vee \sigma(T_1, T_2, \ldots)$
2. $\{V_i\} = \{T_i\}$ and $\mathscr{G} = \sigma(B_s\,;\ s \leq t) \vee \sigma(J_t) \vee \sigma(\Delta_1, \Delta_2, \ldots)$
3. $\{V_i\} = \{\Delta_i, T_i\}$ and $\mathscr{G} = \sigma(B_s\,;\ s \leq t) \vee \sigma(J_t)$

all give

$$A = \{J_t = n\} \in \mathscr{G} \quad \text{and} \quad \Phi(X_t, Y_t) = \varphi(\omega, V_1, \ldots, V_n) \text{ on the set } A$$

where $\varphi : \Omega \times \mathbb{R}^n \to \mathbb{R}$ is measurable w.r.t. $\mathscr{G} \times \mathscr{B}(\mathbb{R}^n)$.

$\rightsquigarrow$ we can use the results in [BBM].

Set $A \in \mathscr{G}$ and $\mathscr{G}_i = \mathscr{G} \vee \sigma(V_k\,;\, k \neq i)$.

Assumption 1 *For any i, $V_i \in L^p$ for all p and the conditional law of V_i given $\mathscr{G}_i$ is absolutely continuous w.r.t. the Lebesgue measure. Moreover, if $p_i = p_i(\omega, \cdot)$ denotes the density, then*

- *for $\omega \in A$, $p_i(\omega, \cdot) \in C^1(B_i(\omega))$, where*

$$B_i(\omega) = \bigcup_{j=1}^{N_i(\omega)} (\alpha_i^j(\omega), \beta_i^j(\omega))$$

  where $-\infty \leq \alpha_i^1(\omega) < \beta_i^1(\omega) < \cdots < \alpha_i^{N_i(\omega)}(\omega) < \beta_i^{N_i(\omega)}(\omega) \leq +\infty$;

- *$\partial_v \ln p_i(\cdot, v) \mathbf{1}_A \in L^p(\Omega, \mathscr{F}, \mathbb{P})$, $\forall p$.*

1. $\{V_i\}_i = \{\Delta_i\}_i$

Assumption 1 becomes:

Assumption A. *$\Delta_i \in L^p$ for all p and its law has a density g w.r.t. the Lebesgue measure. Moreover, $g \in C^1(B)$ where*

$$B = \cup_{i=1}^{N}(a_i, b_i)$$

for suitable $-\infty \leq a_1 < b_1 < \cdots < a_N < b_N \leq +\infty$.

2. $\{V_i\}_i = \{T_i\}_i$

Assumption 1 is fulfilled:

$$p_i(\omega, v) = \frac{1}{T_{i+1}(\omega) - T_{i-1}(\omega)} \mathbf{1}_{v \in \left(T_{i-1}(\omega), T_{i+1}(\omega)\right)} \quad \text{for} \quad \omega \in \{J_t = n\}$$

with the convention $T_0 = 0$ and for $\omega \in \{J_t = n\}$, $T_{n+1} = t$.

3. $\{V_i\}_i = \{\Delta_i, T_i\}_i$

We ask for Assumption A.

The space $\mathcal{S}_{n,k}(A)$ of the simple functionals of order n on A: $F \in \mathcal{S}_{n,k}(A)$ iff

$$F = f(\omega, V_1, \ldots, V_n) \quad \text{for} \quad \omega \in A,$$

where $f : \Omega \times \mathbb{R}^n \to \mathbb{R}$ is $\mathscr{G} \times \mathscr{B}(\mathbb{R}^n)$-measurable and $f(\omega, \cdot) \in C^k(\mathbb{R}^n)$ a.s. on A.

Malliavin derivative: $D : \mathcal{S}_{n,1}(A) \to \mathcal{P}_{n,0}(A)$ is defined as

$$D_i F = \partial_{V_i} f(\cdot, V_1, \ldots, V_n),\ i = 1, \ldots, n.$$

1. $\{V_i\}_i = \{\Delta_i\}_i$: $\mathcal{S}_{n,k}(A) = \mathcal{S}^{\Delta}_{n,k}(\{J_t = n\})$

   R + R(Δ) $\Rightarrow$ $X_t, Y_t \in \mathcal{S}_{n,2}(\{J_t = n\})$ and $\xi_t, \beta_t \in \mathcal{S}_{n,1}(\{J_t = n\})$.

2. $\{V_i\}_i = \{T_i\}_i$: $\mathcal{S}_{n,k}(A) = \mathcal{S}^{T}_{n,k}(\{J_t = n\})$

   R + R(T) $\Rightarrow$ $X_t, Y_t \in \mathcal{S}_{n,2}(\{J_t = n\})$ and $\xi_t, \beta_t \in \mathcal{S}_{n,1}(\{J_t = n\})$.

3. $\{V_i\}_i = \{\Delta_i, T_i\}_i$: $\mathcal{S}_{n,k}(A) = \mathcal{S}^{\Delta}_{n,k}(\{J_t = n\}) \cap \mathcal{S}^{T}_{n,k}(\{J_t = n\})$

   R + R(Δ) + R(T) $\Rightarrow$ $X_t, Y_t \in \mathcal{S}_{n,2}(\{J_t = n\})$, $\xi_t, \beta_t \in \mathcal{S}_{n,1}(\{J_t = n\})$.

The space $\mathcal{P}_{n,k}(A)$ of the simple processes of order n on A: $U \in \mathcal{P}_{n,k}(A)$ iff $U = (U_1, \dots, U_n)$, where $U_i \in \mathcal{S}_{n,k}(A)$.

For $U, Z \in \mathcal{P}_{n,0}(A)$, consider the (random) scalar product

$$\langle U, Z \rangle_\pi = \sum_{i=1}^{n} U_i Z_i \pi_i \qquad \text{on } A,$$

where $\pi_i = \pi_i(\omega, V_i)$ is such that

- $\pi_i : \Omega \times \mathbb{R} \to \mathbb{R}_+$ is $\mathscr{G} \times \mathscr{B}(\mathbb{R})$-measurable;
- for a.e. $\omega \in A$, $\pi_i(\omega, \cdot) \in C^1(\mathbb{R})$;
- $\pi_i(\cdot, V_i)\mathbf{1}_A \in L^p(\Omega, \mathscr{F}, \mathbb{P})$ for any p and $\partial_{V_i}\pi_i(\cdot, V_i)\mathbf{1}_A \in L^{1+\delta}(\Omega, \mathscr{F}, \mathbb{P})$ for some $\delta > 0$

Skorohod integral: $\delta : \mathcal{P}_{n,1}(A) \to \mathcal{S}_{n,0}(A)$ is defined as

$$\delta(U) = \sum_{i=1}^{n} \delta_i(U) \quad \text{where} \quad \delta_i(U) = -\Big(\partial_{V_i}(\pi_i u_i) + \pi_i u_i \partial_{V_i} \ln p_i\Big)(\omega, V_1, \ldots, V_n).$$

1. $\{V_i\}_i = \{\Delta_i\}_i$: $\mathcal{P}_{n,k}(\{J_t = n\}) = \mathcal{P}^{\Delta}_{n,k}(\{J_t = n\})$

$$\delta_i(U) = \delta_i(U_i) = -\Big(\partial_{\Delta_i}(\pi_i u_i) + \pi_i u_i (\ln g)'\Big)(\omega, \Delta_1, \ldots, \Delta_n, T_1, \ldots, T_n)$$

2. $\{V_i\}_i = \{T_i\}_i$: $\mathcal{P}_{n,k}(\{J_t = n\}) = \mathcal{P}^{T}_{n,k}(\{J_t = n\})$

$$\delta_i(U) = \delta_i(U_i) = -\partial_{T_i}(\pi_i u_i)(\omega, \Delta_1, \ldots, \Delta_n, T_1, \ldots, T_n)$$

3. $\{V_i\}_i = \{\Delta_i, T_i\}_i$: $\mathcal{P}_{n,k}(\{J_t = n\}) = \mathcal{P}^{\Delta}_{n,k}(\{J_t = n\}) \times \mathcal{P}^{T}_{n,k}(\{J_t = n\})$

$$\delta_i(U) = \delta_i(U_i) = -\Big(\partial_{\Delta_i}(\pi_i u_i) + \pi_i u_i (\ln g)'\Big)(\omega, \Delta_1, \ldots, \Delta_n, T_1, \ldots, T_n) + \\ -\partial_{T_i}(\pi_i u_i)(\omega, \Delta_1, \ldots, \Delta_n, T_1, \ldots, T_n)$$

For $F, \widehat{F} \in \mathcal{S}_{n,0}(A)$, for $i = 1, \ldots, n$ on A one sets

$$\Gamma_i(F, \widehat{F}) = \sum_{k=1}^{N_i} \Big[(f \widehat{f} \pi_i p_i)(\omega, V_1, \ldots, V_{i-1}, (\beta_i^k)^-, V_{i+1}, \ldots, V_n) + \\ - (f \widehat{f} \pi_i p_i)(\omega, V_1, \ldots, V_{i-1}, (\alpha_i^k)^+, V_{i+1}, \ldots, V_n).$$

Notice that $\Gamma_i(\cdot, \cdot) = 0$ whenever

$$p_i(\omega, \cdot) = 0 \quad \text{on} \quad \partial B_i(\omega) \quad \text{or} \quad \pi_i(\omega, \cdot) = 0 \quad \text{on} \quad \partial B_i(\omega).$$

Border term operator: for $(F, U) \in \mathcal{S}_{n,0}(A) \times \mathcal{P}_{n,0}(A)$, the border term operator is given by

$$[F, U]_\pi = \sum_{i=1}^{n} \Gamma_i(F, U_i) \quad \text{on the set } A.$$

If $\pi_i = 1$ $\forall i$:

1. $\{V_i\}_i = \{\Delta_i\}_i$: if $g = 0$ on ∂B then $[\cdot, \cdot]_\pi = 0$
2. $\{V_i\}_i = \{T_i\}_i$: $[\cdot, \cdot]_\pi$ not null
3. $\{V_i\}_i = \{\Delta_i, T_i\}_i$: if $g = 0$ on ∂B then $[\cdot, \cdot]_\pi$ is not null but the contribution comes only from the T_i's

- Duality relationship

If $F \in \mathcal{S}_{n,1}(A)$ *and* $U \in \mathcal{P}_{n,1}(A)$ *are such that* $\mathbb{E}(|F\delta_i(U)|\mathbf{1}_A) + \mathbb{E}(\pi_i |D_i F U_i| \mathbf{1}_A) < \infty$ *for any* $i = 1, \ldots, n$, *one has*

$$\mathbb{E}\big(\langle DF, U\rangle \mathbf{1}_A\big) = \mathbb{E}\big(F\delta(U)\mathbf{1}_A\big) + \mathbb{E}\big([F,U]_\pi \mathbf{1}_A\big).$$

- Chain rule

Let $F_1, \ldots, F_d \in \mathcal{S}_{n,1}(A)$ *be such that* $D_i F_j \mathbf{1}_A \in L^p$ $\forall p$. *If* $\Phi \in C^1_p(\mathbb{R}^d)$ *then* $\phi(F_1, \ldots, F_d) \in \mathcal{S}_{n,1}(A)$ *and*

$$D_i \Phi(F_1, \ldots, F_d) = \sum_{k=1}^{d} \partial_k \Phi(F_1, \ldots, F_d) D_i F_k, \quad i = 1, \ldots, n.$$

- Malliavin integration by parts formula

Let $F=(F^1,\dots,F^d)$ be such that $F^i\in\mathcal{S}_{n,2}(A)$ and $F^i\in L^p$ for all p. Let $U^1,\dots U^d\in\mathcal{P}_{n,1}(A)$ and let $\sigma_{U,DF}$ be the random matrix defined as

$$\sigma^{\ell k}_{U,DF}=\langle U^\ell,DF^k\rangle_\pi,\quad \ell,k=1,\dots,d.$$

Suppose that $\sigma_{U,DF}$ is a.s. invertible on A and setting $\widehat\sigma_{U,DF}=\widehat\sigma^{-1}_{U,DF}$, assume that $\widehat\sigma^{\ell k}_{U,DF}\in\mathcal{S}_{n,1}(A)$, for all ℓ,k. Let $G\in\mathcal{S}_{n,1}(A)$ and suppose that there exists $\eta>0$ such that

$$\widehat\sigma^{\ell k}_{U,DF}U^k_iG\mathbf{1}_A\in L^{1+\eta}\quad\text{and}\quad\delta_i(\widehat\sigma^{\ell k}_{U,DF}U^kG)\mathbf{1}_A\in L^{1+\eta},\quad\forall\ell,k,i. \tag{4}$$

Then, for any $\Phi\in C^1_p(\mathbb{R}^d)$ and $\ell=1,\dots,d$ one has

$$\mathbb{E}\big(\partial_\ell\Phi(F)G\big)=\mathbb{E}\Big(\Phi(F)\delta\Big(\sum_{k=1}^d\widehat\sigma^{\ell k}_{U,DF}U^kG\Big)\mathbf{1}_A\Big)+\mathbb{E}\Big(\Big[\Phi(F),\widehat\sigma^{\ell k}_{U,DF}U^kG\Big]_\pi\mathbf{1}_A\Big).$$

- If $U^k_i\mathbf{1}_A,G\mathbf{1}_A\in L^p$ $\forall p$ and if $|\det\sigma_{U,DF}|^{-1}\mathbf{1}_A\in L^p$ $\forall p$, then (4) holds.

- The case $U^k=DF^k$ gives $\sigma_{U,DF}=\sigma_F$, i.e. the Malliavin covariance matrix, and the above requirement is the classical non-degeneracy condition.

- Representation formula for the delta

Theorem 1 *Set* $Z_t = (X_t, Y_t)$*. Let* $U^1, U^2 \in \mathcal{P}_{n,1}(\{J_t = n\})$ *and for* $\ell, k = 1, 2$*, let*

$$\sigma^{\ell k}_{U,DZ_t} = \langle U^\ell, DF^k \rangle_\pi .$$

Suppose that σ_{U,DZ_t} *is a.s. invertible on* $\{J_t = n\}$*, with inverse* $\widehat{\sigma}_{U,DZ_t}$*, and assume that* $\widehat{\sigma}^{\ell k}_{U,DZ_t} \in \mathcal{S}_{n,1}(\{J_t = n\})$*. Set* $G = (\xi_t, \beta_t)$ *and suppose there exists* $\eta > 0$ *such that*

$$\widehat{\sigma}^{\ell k}_{U,DF} U^k_i G^\ell \mathbf{1}_{\{J_t = n\}} \in L^{1+\eta} \quad \text{and} \quad \delta_i(\widehat{\sigma}^{\ell k}_{U,DF} U^k G^\ell) \mathbf{1}_{\{J_t = n\}} \in L^{1+\eta}, \quad \forall \ell, k, i. \tag{5}$$

Then, for any Φ *with polynomial growth one has*

$$\partial_x \mathbb{E}\big(\Phi(X_t, Y_t) \mathbf{1}_{\{J_t = n\}}\big) = \mathbb{E}\Big(\Phi(X_t, Y_t) \delta(W) \mathbf{1}_{\{J_t = n\}}\Big) + \mathbb{E}\Big([\Phi(X_t, Y_t), W]_\pi \mathbf{1}_{\{J_t = n\}}\Big)$$

where

$$W = \sum_{k=1}^{2} \widehat{\sigma}^{1k}_{U,DZ_t} U^k \xi_t + \sum_{k=1}^{2} \widehat{\sigma}^{2k}_{U,DZ_t} U^k \beta_t .$$

(5) is a *weak non degenerate condition*. It holds if the (standard) non degenerate condition, i.e. $|\det \sigma_{U,DZ_t}|^{-1} \in L^p \ \forall p$, is fulfilled.

- Formulas in practice

We always refer to $\pi_i = 1$, $\forall i$ and $g = 0$ on ∂B. And we set

$$\mathscr{A}_n^J = \Big\{ \alpha \in \mathbb{R}^n \text{ such that } \sum_{i=1}^n \alpha_i = 0 \Big\}.$$

1. $\{V_i\}_i = \{\Delta_i\}_i$: we assume R + R($\Delta$)

Here, $[\cdot,\cdot]_\pi = 0$.

We set $\partial_a c_i = \partial_a c(T_i, \Delta_i, X_{T_i^-})$ and for $\alpha \in \mathscr{A}_n^J$, we consider

$$U_i^1 = \xi_{T_i}(\partial_a c_i)^{-1}\alpha_i, \qquad U_i^2 = \xi_{T_i}(\partial_a c_i)^{-1}, \quad i = 1, \dots, n$$

This gives

$$\sigma_{U,DZ_t} = \begin{pmatrix} 0 & -\sum_{i=1}^n \beta_{T_i}\alpha_i \\ n\xi_t & n\beta_t - \sum_{i=1}^n \beta_{T_i} \end{pmatrix} \text{ and } W_i = \frac{\xi_{T_i}(\partial_a c_i)^{-1}}{n}\Big(1 - \frac{\sum_{j=1}^n \beta_{T_j}}{\sum_{j=1}^n \beta_{T_j}\alpha_j}\,a_i\Big),$$

$i = 1, \dots, n$,

One has

Proposition *If* $A_k = \sum_{i=1}^{k} \alpha_i > 0$ *for any* $k = 1, \ldots, n-1$, *then* $\mathbf{1}_{\{J_T = n\}} | \sum_{i=1}^{n} \beta_{T_i} \alpha_i |^{-1} \in L^p$ *for* $p < n/2$.

Then, the weak non degenerate condition holds if $n > 4$. However, from the numerical point of view, the formula efficiently work for any $n \geq 2$ if the intensity λ is not too small.

In practice, we take $\alpha_i = 1$ for any $i < n$ and $\alpha_n = 1 - n$, so that $A_k = k > 0$ for any $k = 1, \ldots n - 1$.

2. $\{V_i\}_i = \{T_i\}_i$: we assume R + R(T) + ($\sigma = 0$)

In [BBM]: for $\gamma \in (0,1)$,

$$\pi_i(\omega, r) = (r - T_{i-1})^\gamma (T_{i+1} - r)^\gamma, \quad r \in (T_{i-1}, T_{i+1}),$$

so that $[\cdot,\cdot]_\pi = 0$. But here, $[\cdot,\cdot]_\pi \neq 0$.

We set $q_i = q(T_i, \Delta_i, X_{T_i^-})$ and for $\alpha \in \mathscr{A}_n^J$, we consider

$$U_i^1 = \xi_{T_i}(q_i)^{-1}\alpha_i, \qquad U_i^2 = \xi_{T_i}(q_i)^{-1}, \quad i = 1, \ldots, n$$

This gives

$$\sigma_{U,DZ_t} = \begin{pmatrix} 0 & -\sum_{i=1}^n \beta_{T_i}\alpha_i \\ n\xi_t & n\beta_t - \sum_{i=1}^n \beta_{T_i} \end{pmatrix} \text{ and } W_i = \frac{\xi_{T_i} q_i^{-1}}{n}\Big(1 - \frac{\sum_{j=1}^n \beta_{T_j}}{\sum_{j=1}^n \beta_{T_j}\alpha_j}\,\alpha_i\Big),$$

$i = 1, \ldots, n$. Then, we obtain the same matrix as in 1. and the same arguments for the study of the weak non degeneracy condition can be used.

3. $\{V_i\}_i = \{\Delta_i, T_i\}_i$: we assume R + R($\Delta$) + R($T$) + ($\sigma = 0$)

Here, $[\cdot,\cdot]_\pi \neq 0$ (and it acts on the components referring to the T_i's).

As for U, we mix the choices already done. For example, for $\alpha \in \mathscr{A}_n^J$ we consider

$$\begin{array}{ll} U_i^1 = \xi_{T_i}(\partial_a c_i)^{-1} a_i, & U_{i+n}^1 = 0 \\ U_i^2 = 0 & U_{i+n}^2 = \xi_{T_i}(q_i)^{-1} \end{array}$$

for $i = 1, \ldots, n$. This gives again the same generalized Malliavin covariance matrix, so that setting $\alpha_i = 1$ for any $i < n$ and $\alpha_n = 1 - n$, one has $A_k = k > 0$ for any $k = 1, \ldots n-1$. Moreover, for $i = 1, \ldots, n$,

$$W_i = \frac{\sum_{j=1}^n \beta_{T_j}}{n \sum_{j=1}^n \beta_{T_j} a_j} \cdot \xi_{T_i}(\partial_a c_i)^{-1} a_i, \qquad W_{i+n} = \frac{1}{n} \cdot \xi_{T_i} q_i^{-1}$$

5. Weights from the Brownian and the jump noise

We can compose the Brownian Malliavin calculus with the Malliavin calculus in the direction of the jump amplitudes noise, and we obtain similar IBP formulas.

- Formulas in practice

We mix the processes already separately found: for $\alpha \in \mathscr{A}_t^W$, we consider e.g.

$$\begin{array}{ll} U^1_{0,s} = \xi_s \sigma^{-1}(X_s)\alpha_s, & U^1_i = 0 \\ U^2_{0,s} = 0 & U^2_i = \xi_{T_i}(\partial_a c_i)^{-1} \end{array}$$

The associated generalized Malliavin covariance matrix satisfies the non degeneracy condition and

$$W_{0,s} = \frac{\sum_{i=1}^n \beta_{T_i}}{n \int_0^t \beta_r \alpha_r dr} \cdot \xi_u \sigma^{-1}(X_u)\alpha_u, \qquad W_i = \frac{1}{n} \cdot \xi_{T_i}(\partial_a c_i)^{-1} \quad \text{as} \quad i = 1, \ldots, n$$

The formula is then

$$\partial_x \mathbb{E}\big(\Phi(X_t, Y_t)\mathbf{1}_{\{J_t=n\}}\big) = \mathbb{E}\Big(\Phi(X_t, Y_t)\Big[\delta_0(W_0) + \sum_{i=1}^n \delta_i(W_i)\Big]\mathbf{1}_{\{J_t=n\}}\Big).$$

6. Numerical results

Tested models:

- ◇ Black& Scholes model with jumps: $b(x) = bx$, $\sigma(x) = \sigma x$ $c(a,x) = \alpha a x$;

- ◇ Stein&Stein model with jumps:

$$X_t = x + \int_0^t \mu X_r dr + \int_0^t \sigma_r X_r \Big(\sqrt{1-\rho^2} dW_r^1 + \rho\, dW_r^2\Big) + \alpha \sum_{i=1}^{J_t} \Delta_i X_{T_i-}$$

$$\sigma_t = y + \int_0^t k(\theta - \sigma_r) dr + \int_0^t \beta dW_r^2$$

  Remark: using the noise from W^1, nothing changes.

- ◇ Ornstein&Uhlembek model with jumps: $b(x) = b \cdot (\theta - x)$, $\sigma(x) = \sigma$, $c(a,x) \equiv c(a) = \alpha a$.

  Special case: here $X_t = xe^{-bt} + \theta(1 - e^{-bt}) + \sigma e^{-bt} \int_0^t e^{br} dW_r + \alpha \sum_{i=1}^{J_t} e^{-b(t-T_i)} \Delta_i$ and one can compose all the random noises.

- ◇ CIR model with jumps: $b(x) = \nu - \eta x$, $\sigma(x) = \sigma\sqrt{x}$, $c(a,x) \equiv c(a) = \alpha(1+a)$.

Tested payoffs:

- floating Asian call and put option:

$$f(X_T, Y_T) = (X_T - Y_T/T)_+ \text{ and } f(X_T, Y_T) = (Y_T/T - X_T)_+;$$

- (standard) Asian digital, call and put option:

$$f(X_T, Y_T) = \mathbf{1}_{Y_T/T>K},\ f(X_T, Y_T) = (Y_T/T - K)_+ \text{ and } f(X_T, Y_T) = (K - Y_T/T)_+;$$

- standard digital option:

$$f(X_T, Y_T) = \mathbf{1}_{X_T>K}.$$

- ✓ A benchmark value for the delta is obtained using the finite difference method with $250,000$ simulations ($\delta = 10^{-3}x$).

- ✓ When a discretization of the time interval is needed for the simulation, we split the time interval in 100 subintervals.

- ✓ Unless specified, we set

  [jump amplitudes] $1 + \Delta_i \stackrel{\mathscr{L}}{=} e^{m+\varrho\mathcal{N}_i}$, with $m = 0$, $\varrho = 0.05$ and $\mathcal{N}_i \sim \mathrm{N}(0,1)$;

  [Poisson intensity] $\lambda = 5$ unless specified (we test also small value of λ);

  [maturity time] $T = 5$.

  [starting underlying asset price] $x = 100$.

- ✓ We consider also a variance reduction technique based on localization functions, build as a natural generalization of the ones in [BavM].

Remark

We numerically compute

$$\partial_x \mathbb{E}(\Phi(X_t, Y_t)) = \sum_{n \geq 0} \partial_x \mathbb{E}(\Phi(X_t, Y_t) \mathbf{1}_{\{J_t = n\}}).$$

trough a Monte Carlo procedure based on the representation of each term by Malliavin weights. But a special attention must be given to the cases $n = 0, 1$

1. Case $n = 0$

$*$ $\sigma = 0$: (X_t, Y_t) is deterministic on $\{J_t = 0\}$, so if Φ is not smooth we actually compute $\partial_x \mathbb{E}(\Phi(X_t, Y_t) \mathbf{1}_{\{J_t \geq 1\}})$.

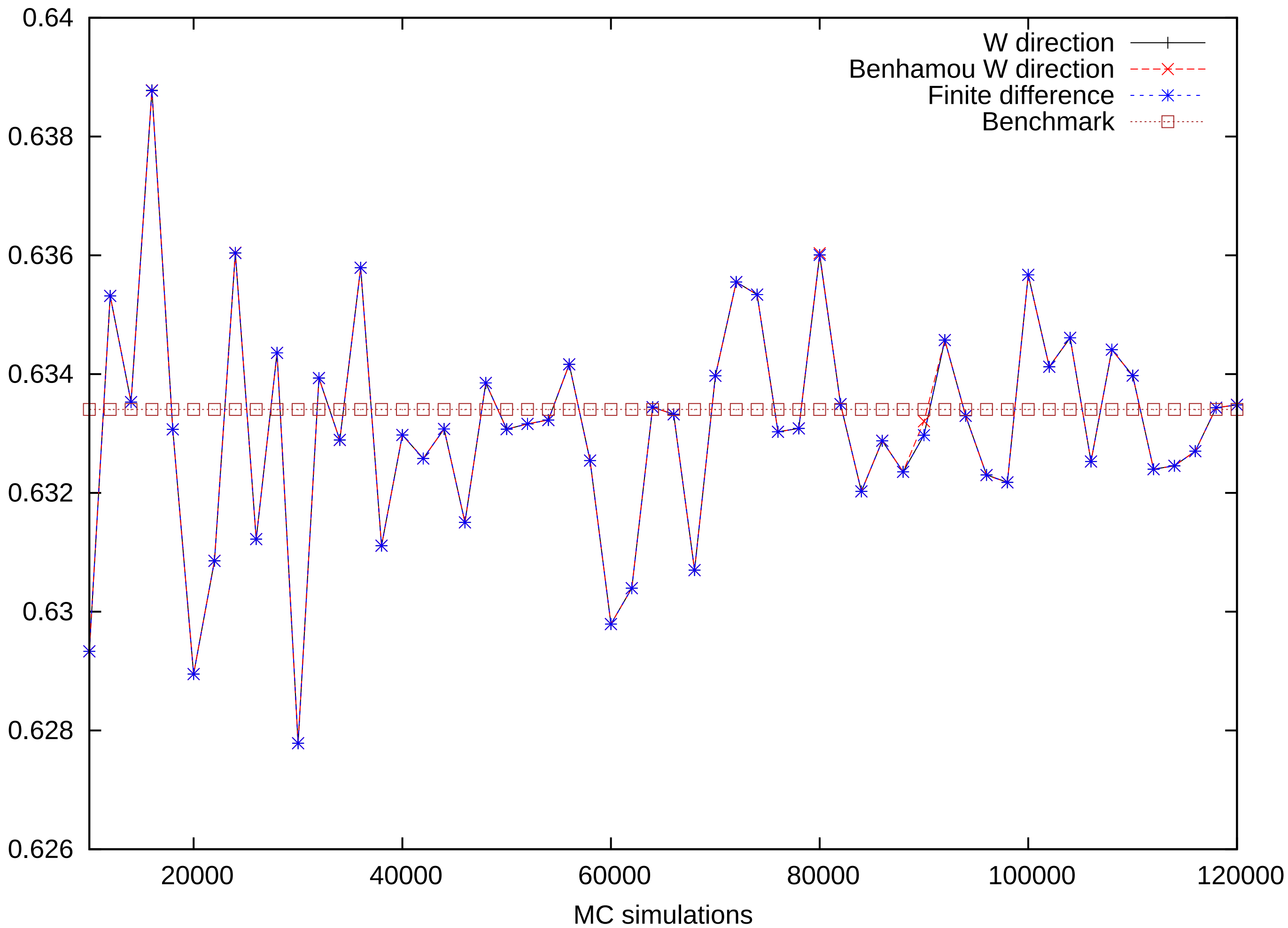
$*$ $\sigma \neq 0$: no jump noise can be used and the term $\partial_x \mathbb{E}(\Phi(X_t, Y_t) \mathbf{1}_{\{J_t = 0\}})$ will be computed always by means of the weight coming from the Brownian noise.

2. Case $n = 1$

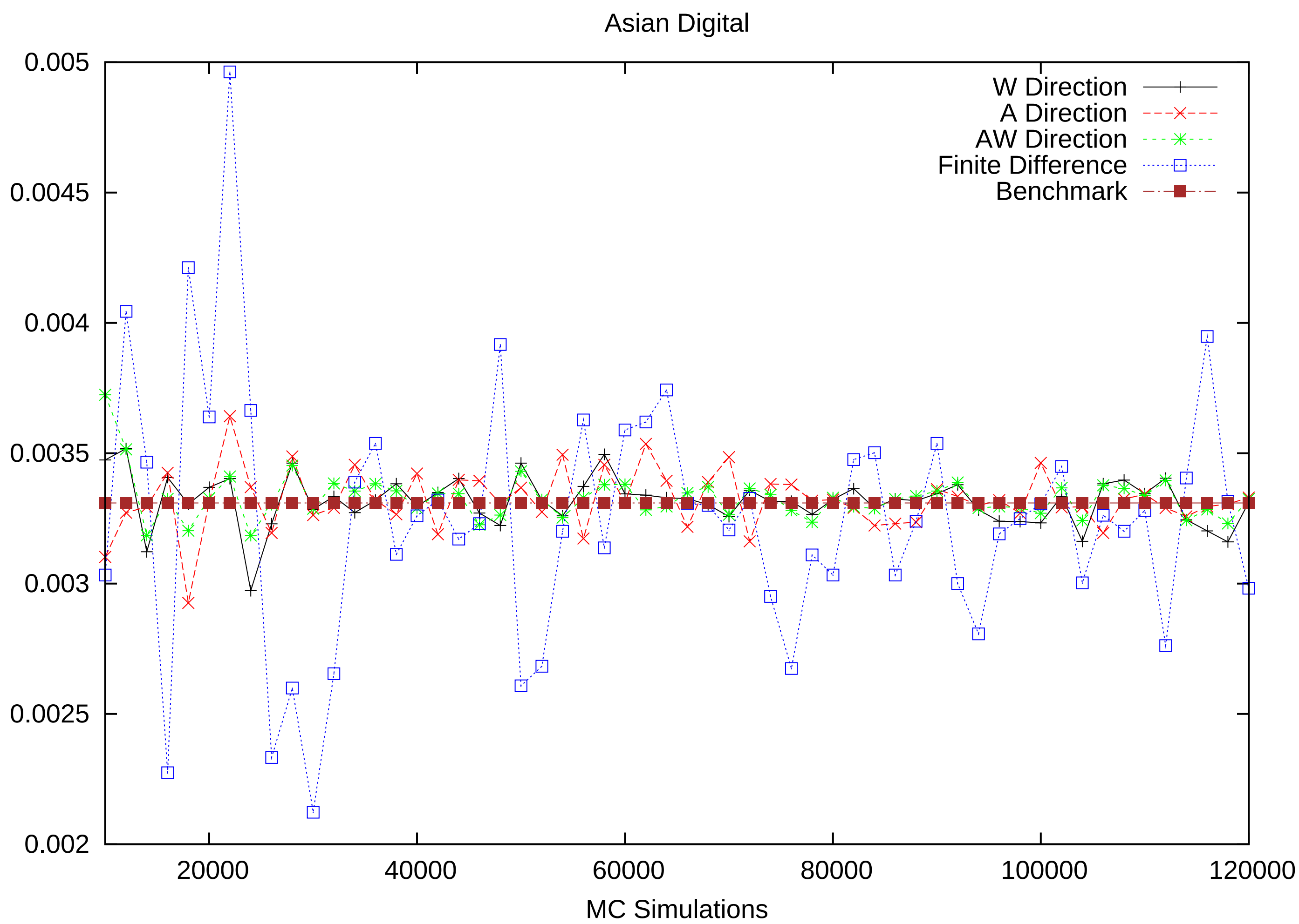
When testing the results from the jump amplitudes noise, on the set $\{J_t = 1\}$ only Δ_1 is available. But, this is a case when the generalized Malliavin matrix is always null. So, the term $\partial_x \mathbb{E}(\Phi(X_t, Y_t) \mathbf{1}_{\{J_t = 1\}})$ is computed by adding the noises T_1 or W (depending on if $\sigma = 0$ or $\sigma \neq 0$ respectively).

Similarly for the jump times noise.

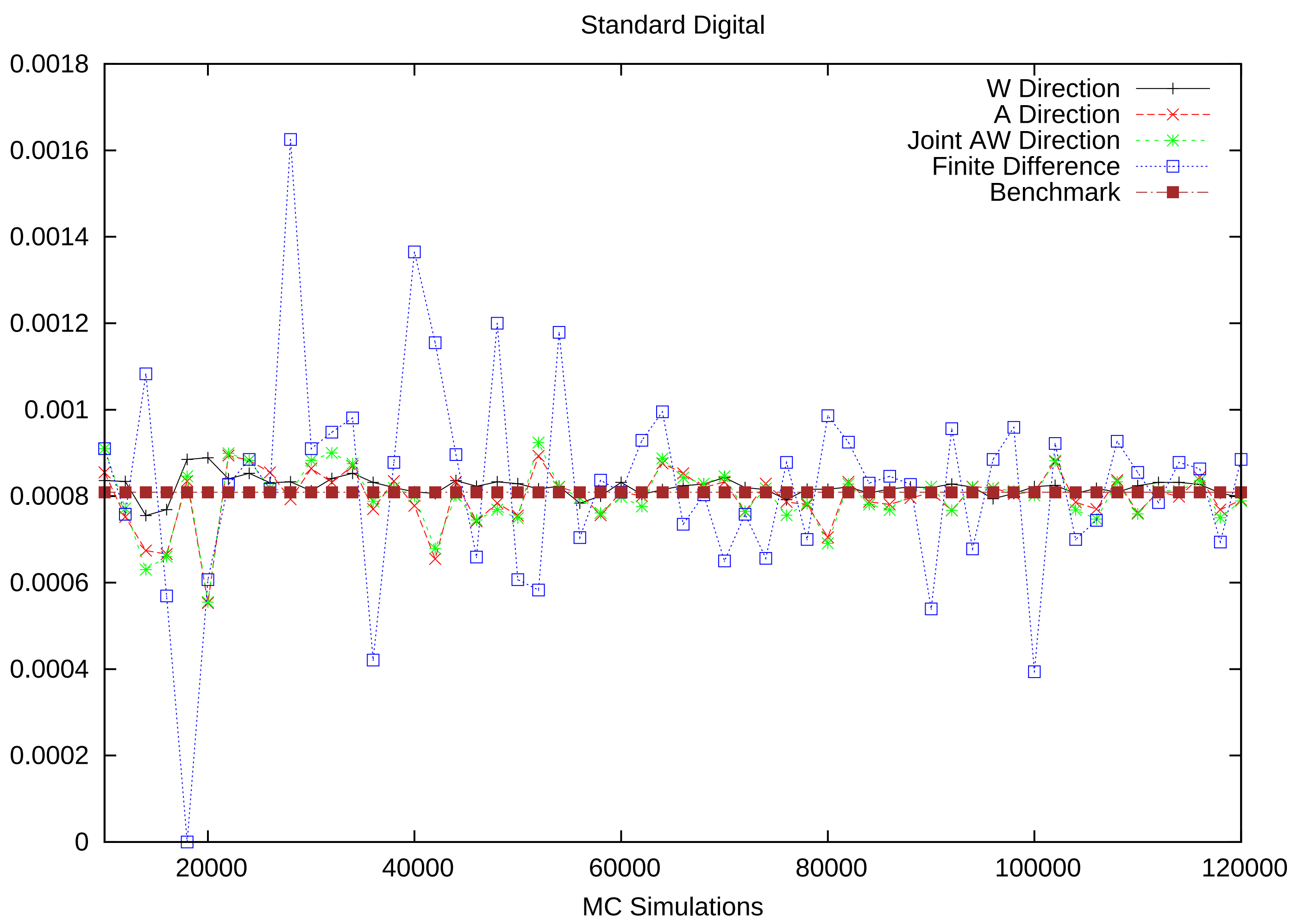
Black-Scholes model, Asian floating option. Comparison with Benhamou's weight. Localized formulas.



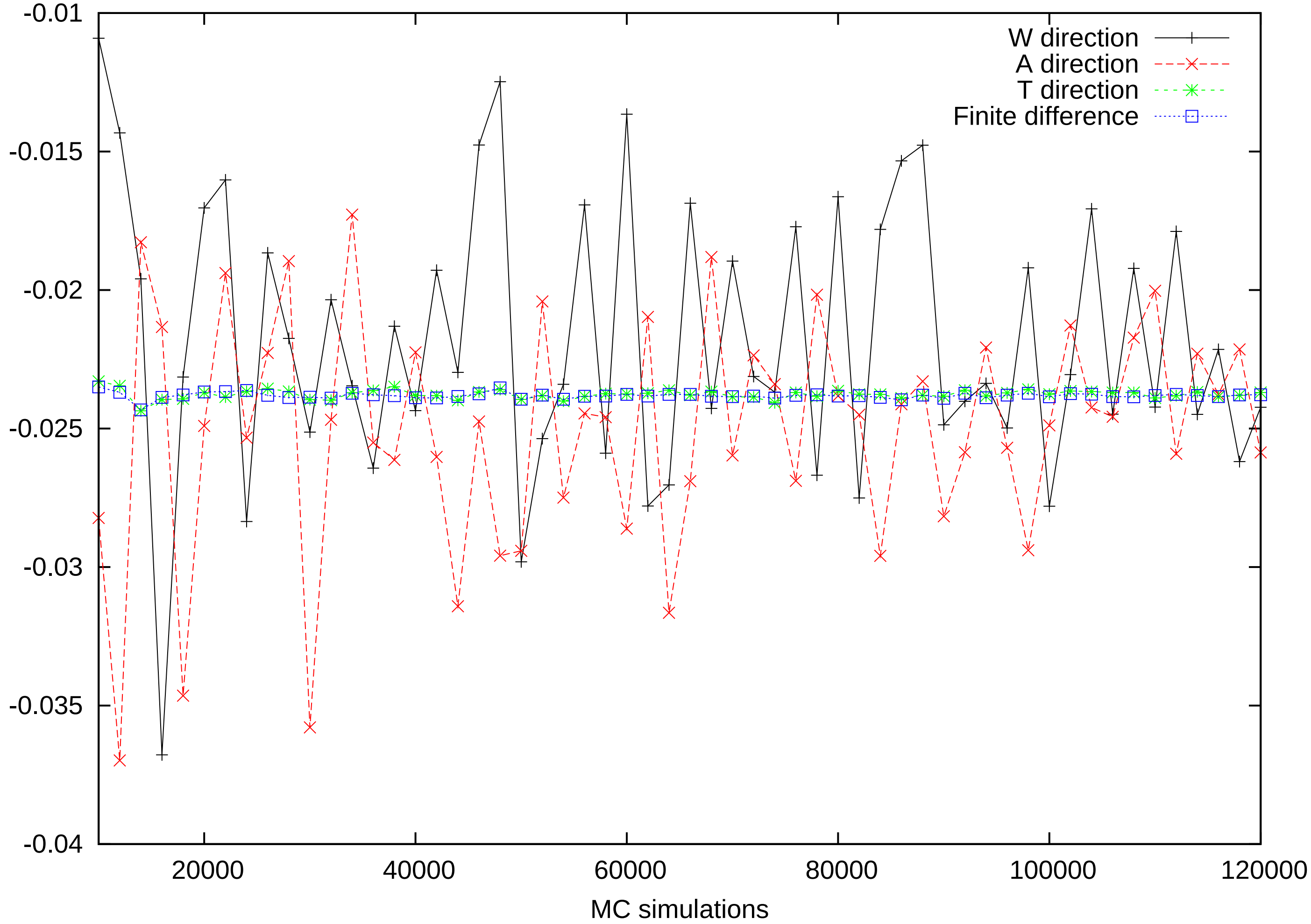
Black-Scholes model, Asian digital option. Non localized formulas.



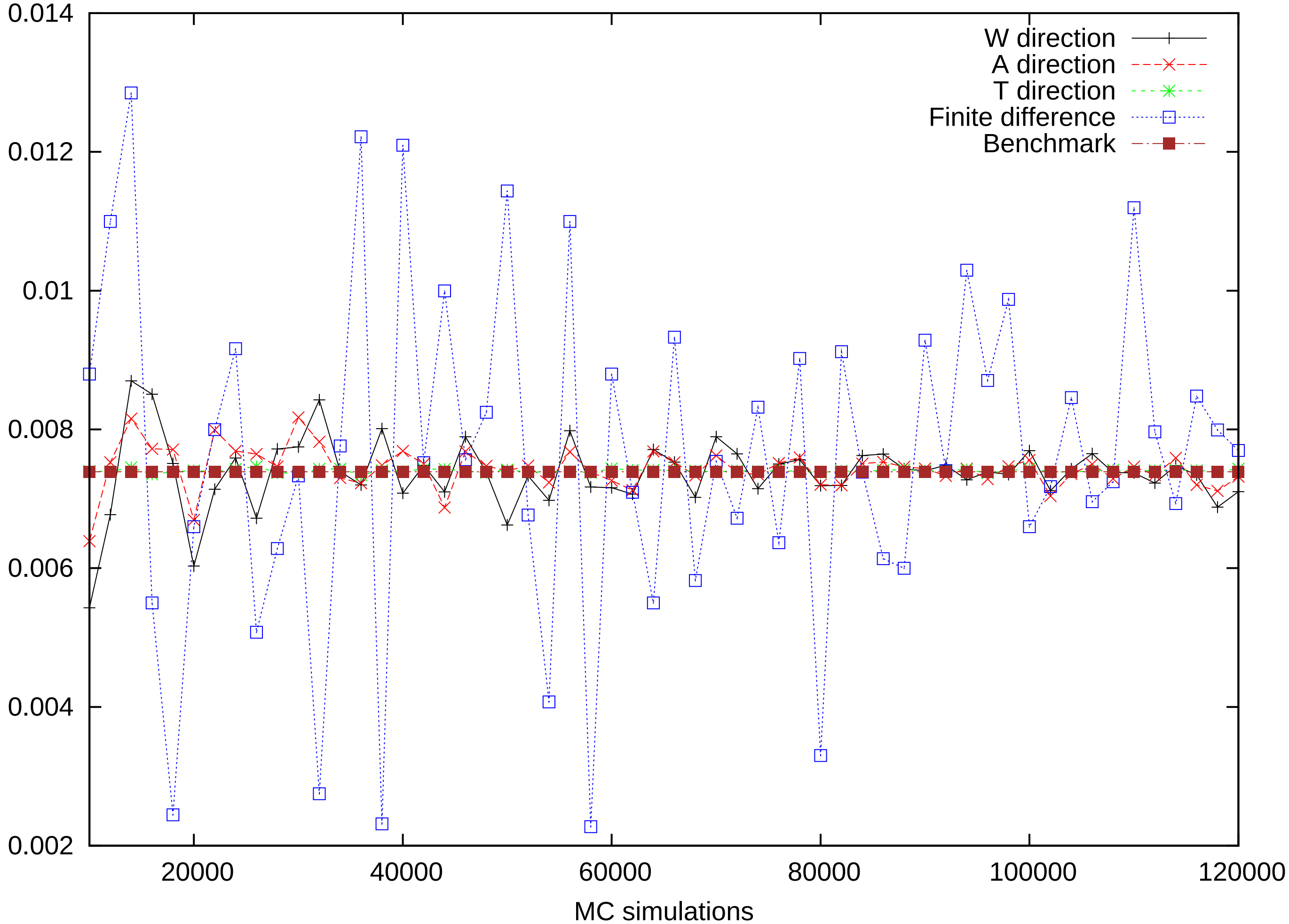
Stein& Stein model, standard digital option. Non localized formulas



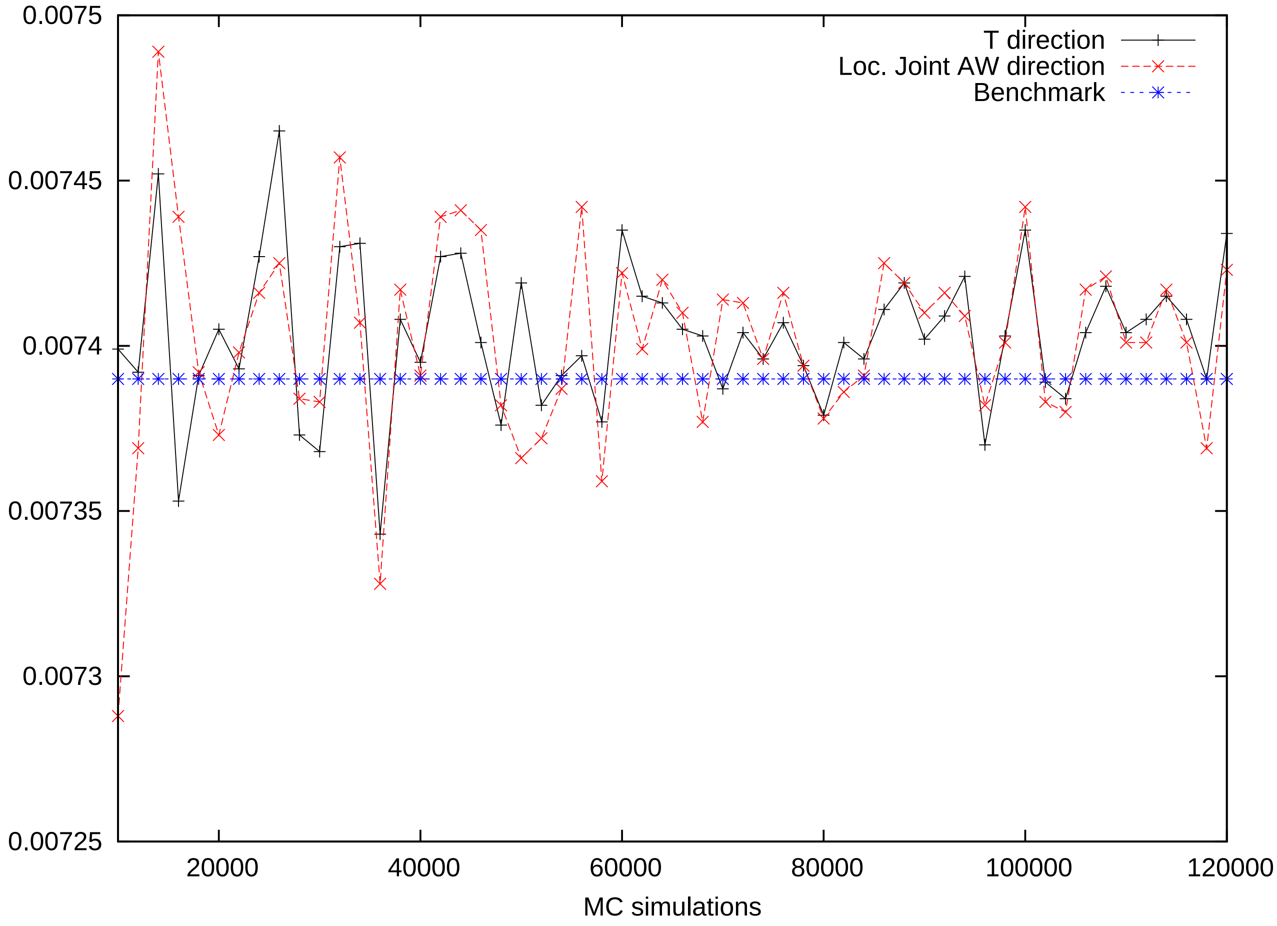
Ornstein-Uhlembeck model, floating Asian option. Non localized formulas.



Ornstein-Uhlembeck model, standard digital option. Non localized formulas.



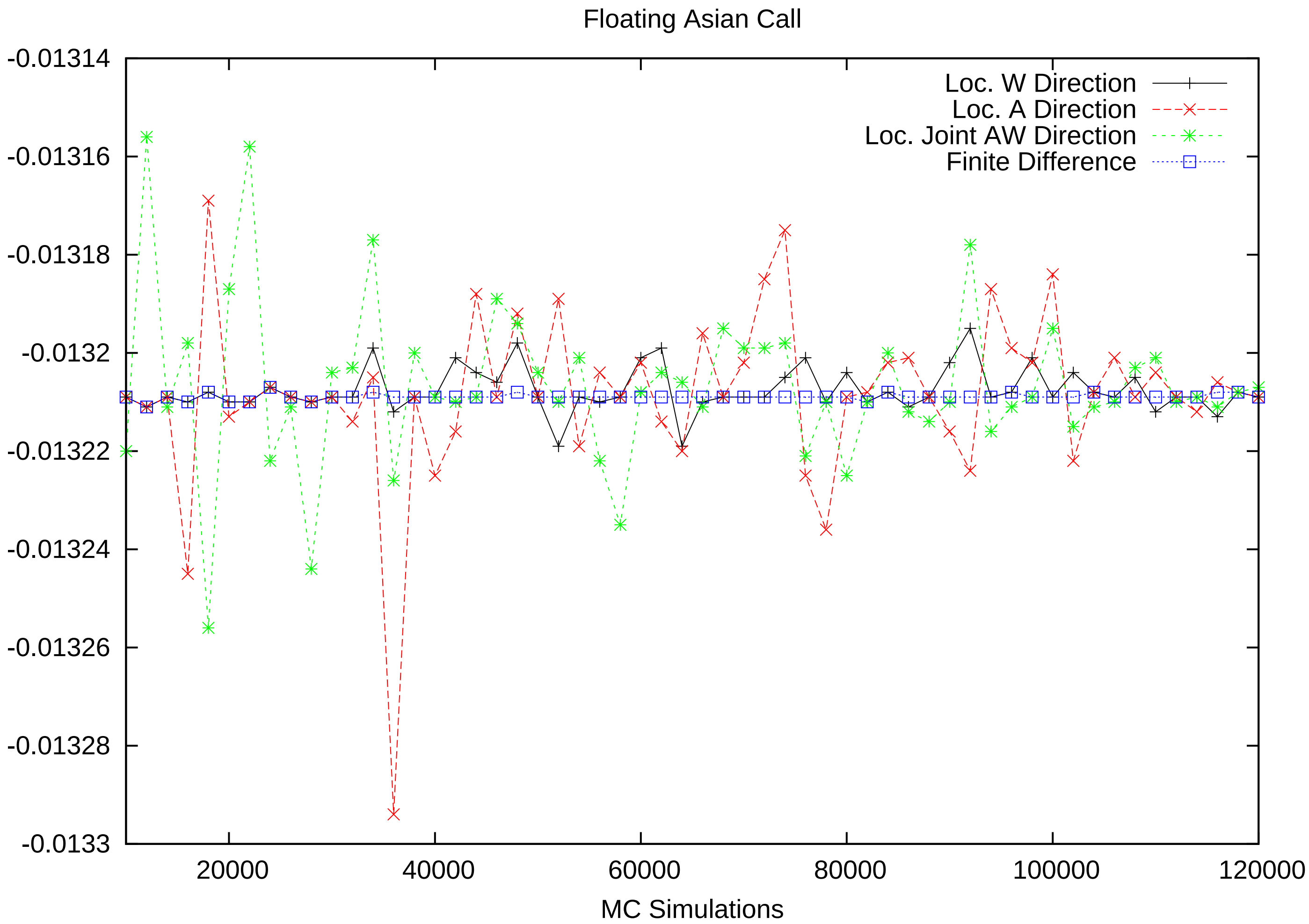
Ornstein-Uhlembeck model, standard digital option. Non localized formulas from the jump times and localized formulas from the joint Brownian and jump amplitudes noises.



Ornstein-Uhlembeck model. Comparison between the variance from our weight and from the weight by Bally, Bavouzet-Morel and Messaoud w.r.t. the jump times. Here, $\Delta_i \sim \mathrm{N}(0,1)$. Recall: α is such that $c(a,x) = \alpha a$.

α	Var MallJT	Var MallJT in [BBM]
15.8114	0.000180	0.028512
16.6667	0.000222	0.041721
17.6777	0.000277	0.040069
18.8982	0.000341	0.041013
20.4124	0.000422	0.043306
22.3607	0.000535	0.040048
25.0000	0.000685	0.040713
28.8675	0.000902	0.036272
35.3553	0.001224	0.034315
50.0000	0.001749	0.033329

CIR model, floating Asian call option. Localized formulas.



CIR model, standard digital option. Non localized formulas.