Applications of Malliavin Calculus in Statistical Inference

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Consider a probability space $(\Omega, \mathcal{F}, P)$ and a Gaussian subspace $\mathcal{H}$ of $L^2(\Omega, \mathcal{F}, P)$ whose elements are zero-mean Gaussian random variables. Let $\mathfrak{H}$ be a separable Hilbert space with scalar product denoted by $\langle \cdot, \cdot \rangle_{\mathfrak{H}}$ and norm $|| \cdot ||_{\mathfrak{H}}$, we will assume there is an isometry

$$\begin{aligned} W : \mathfrak{H} &\rightarrow \mathcal{H} \\ h &\mapsto W(h) \end{aligned}$$

that is

$$E(W(h_1)W(h_2)) = \langle h_1, h_2 \rangle_{\mathfrak{H}},$$

Let $\mathcal{S}$ be the class of smooth random variables $T(W(h_1), ..., W(h_n))$ such that T and all its derivatives have polynomial growth. Given $T \in \mathcal{S}$ we can define its differential as

$$DT = \sum_{i=1}^{n} \partial_i T(W(h_1), W(h_2), ..., W(h_n))h_i.$$

DT can be seen as a random variable with values in $\mathfrak{H}$.

Then we can define the derivative operator as

$$\begin{array}{rcl} D : \mathbb{D}^{1,2} \subseteq L^2(\Omega, \mathbb{R}) & \longrightarrow & L^2(\Omega, \mathfrak{H}) \\ T & \mapsto & DT. \end{array}$$

where $\mathbb{D}^{1,2}$ is the closure of the class of smooth random variables with respect to the norm

$$||T||_{1,2} = \left(E(|T|^2) + E(||DT||^2_{\mathfrak{H}}) \right)^{1/2}$$

Let u be an element of $L^2(\Omega, \mathfrak{H})$ and assume there is an element $\delta(u)$ belonging to $L^2(\Omega)$ and such that

$$E(\langle DT, u\rangle_{\mathfrak{H}}) = E(T\delta(u))$$

for any $T \in \mathbb{D}^{1,2}$, then we say that u belongs to the domain of δ (denoted by $dom(\delta)$) and that δ is the adjoint operator of D.

In the following results we assume that our observations are expressed as the measurable map

$$\begin{aligned} X : \Omega \times \Theta &\rightarrow \mathbb{R}^n \\ (\omega, \theta) &\mapsto x = X(\omega, \theta), \end{aligned}$$

Θ an open subset of $\mathbb{R}$ with the Borelian σ-field and the σ-field in Ω is the σ-field generated by $\mathcal{H}$.

Theorem

Let $X_j \in \mathbb{D}^{1,2}, j = 1, ..., n$ and Z be a random variable with values in $\mathfrak{H}$, in the domain of δ, such that

$$\langle Z, DX_j \rangle_{\mathfrak{H}} = \partial_\theta X_j. \tag{1}$$

assume that

i) *X has density $p(x;\theta) \in C^1$ as function of θ with support, $supp(X)$, independent of θ,*

ii) *Any smooth statistic with compact support in the interior of $supp(X)$ is regular: $\partial_\theta \int_{\mathbb{R}^n} T(x)p(x;\theta)dx = \int_{\mathbb{R}^n} T(x)\partial_\theta p(x;\theta)dx$, for all $\theta \in \Theta$,*

then

$$E(\delta(Z)|X) = \partial_\theta \log p(X;\theta), \text{ a.s. and for all } \theta \in \Theta.$$

Proof.

$$\partial_\theta E_\theta(T(X)) = \sum_{k=1}^{n} E(\partial_{x_k} T(X)\partial_\theta X_k).$$

If we have Z with

$$\langle Z, DX_j\rangle_{\mathfrak{H}} = \partial_\theta X_j$$

then

$$\partial_\theta E(T(X)) = E(\partial_{x_k} T(X)\langle Z, DX_k\rangle_{\mathfrak{H}}).$$

By the chain rule for the derivative operator D,

$$DT(X) = \partial_{x_k} T(X) DX_k,$$

then

$$\begin{aligned}\partial_\theta E(T(X)) &= E(\langle Z, DT(X)\rangle_{\mathfrak{H}}) \\ &= E(T(X)\delta(Z)).\end{aligned}$$

Proof (Cont.) Next, we have

$$\begin{aligned} E(T(X)\delta(Z)) &= \partial_\theta E(T(X)) \\ &= \partial_\theta \int_{\mathbb{R}^n} T(x)p(x;\theta)dx \\ &= \int_{\mathbb{R}^n} T(x)\partial_\theta p(x;\theta)dx \\ &= \int_{\mathbb{R}^n} T(x)\partial_\theta \log p(x;\theta)p(x;\theta)dx \\ &= E(T(X)\partial_\theta \log p(X;\theta)). \end{aligned}$$

Therefore

$$E((\delta(Z) - \partial_\theta \log p(X;\theta))T(X)) = 0.$$

The next goal is to provide a somewhat standard way of finding Z. For example, if there exists U, an n-dimensional random vector with values on H such that

$$\langle U_k, DX_j \rangle_{\mathfrak{H}} = \delta_{kj}$$

where δ_{kj} is Kronecker's delta then

$$Z = \sum_{k=1}^{n} U_k \partial_\theta X_k$$

verifies condition (1) if $Z \in dom(\delta)$. In particular, if

$$(A_{kj}) = (\langle DX_k, DX_j \rangle_{\mathfrak{H}})^{-1}$$

is well defined then we can take

$$U_k = \sum_{j=1}^{n} A_{kj} DX_j.$$

Example. Let $X^{(n)} = (X_{t_1}, X_{t_2}, ..., X_{t_n})$, $0 < t_1 < t_2 ... < t_n$, be observations of the Ornstein-Uhlenbeck process

$$dX_t = -\theta X_t dt + dB_t, t \geq 0, \quad X_0 = 0.$$

Or, by integrating,

$$X_t = \int_0^t e^{-\theta(t-s)} dB_s.$$

Where $B_t, t \geq 0$ is a standard Brownian motion. Let $\mathcal{H}$ be the closed linear space generated by the random variables $\{B_t, 0 \leq t \leq T\}$ and $\mathfrak{H} = L^2([0, T], dx)$. Then the map

$$\begin{aligned} W : \mathfrak{H} &\rightarrow \mathcal{H} \\ \mathbf{1}_{[0,t]}(\cdot) &\mapsto W(\mathbf{1}_{[0,t]}) \equiv \int_0^T \mathbf{1}_{[0,t]}(s) dB_s = B_t \end{aligned}$$

defines a linear isometry, and $W(h)$ is the stochastic integral of the function h. Consequently

$$DX_t = e^{-\theta(t-\cdot)} \mathbf{1}_{[0,t]}(\cdot).$$

We have

$$\begin{aligned}\partial_\theta X_t &= -\int_0^t e^{-\theta(t-s)} X_s ds \\ &= -\int_0^T X_s D_s X_t ds = -\langle X, DX_t \rangle_{\mathfrak{H}},\end{aligned}$$

where we write $DX_t(s) = D_s X_t$. Then

$$\begin{aligned}\partial_\theta \log p(X^{(n)};\theta) &= -E(\delta(X)|X^{(n)}) = -E(\int_0^T X_s dB_s | X^{(n)}) \\ &= -E(\int_0^T X_s dX_s + \theta \int_0^T X_s^2 ds | X^{(n)})\end{aligned}$$

In particular the maximum likelihood estimator of θ is given by

$$\hat{\theta} = -\frac{E(\int_0^T X_s dX_s | X^{(n)})}{E(\int_0^T X_s^2 ds | X^{(n)})}.$$

Let $X^{(n)} = (X_{t_1}, X_{t_2}, ..., X_{t_n})$, $0 = t_0 < t_1 < t_2 ... < t_n$, where

$$X_t = x + \int_0^t b_s(\theta, X_s) ds + \int_0^t \sigma_s(\theta, X_s) dB_s.$$

and B is a standard Brownian motion.

$$DX_t = \partial_x X_t \left(\partial_x X_\cdot\right)^{-1} \sigma_\cdot(\theta, X_\cdot) \mathbf{1}_{[0,t]}(\cdot),$$

Define

$$\beta_t = \left(\partial_x X_t\right)^{-1} \partial_\theta X_t$$

and

$$\tilde{\beta}_t = \partial_x X_t \left(\sigma_t(\theta, X_t)\right)^{-1} \sum_{i=1}^{n} a(t)(\beta_{t_i} - \beta_{t_{i-1}}) \mathbf{1}_{\{t_{i-1} \le t < t_i\}},$$

where

$$a \in L^2[0, T], \int_{t_{i-1}}^{t_i} a(t) dt = 1, i = 1, ..., n.$$

Then, it is easy to see that

$$\partial_\theta X_{t_i} = \langle \tilde{\beta}, DX_{t_i} \rangle_{\mathfrak{H}}.$$

Consequently the score function is given by

$$E(\delta(\tilde{\beta})|X^{(n)}),$$

where

$$\begin{aligned}\delta(\tilde{\beta}) &= \sum_{i=1}^{n} (\beta_{t_i} - \beta_{t_{i-1}}) \int_{t_{i-1}}^{t_i} a(t) \partial_x X_t \left(\sigma_t(\theta, X_t)\right)^{-1} dB_t \\ &- \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} a(t) D_t \beta_{t_i} \partial_x X_t \left(\sigma_t(\theta, X_t)\right)^{-1} dt\end{aligned}$$

Let $\Delta t_i := t_i - t_{i-1} = 1/n$, and consider n observations, we obtain

$$E(\delta(\tilde{\beta})|X^{(n)}) = n\sum_{i=1}^{n}\left\{\frac{\partial_\theta \sigma_{t_{i-1}}}{\sigma^3_{t_{i-1}}}(\Delta X_{t_i})^2 - \frac{1}{n}\frac{\partial_\theta \sigma_{t_{i-1}}}{\sigma_{t_{i-1}}}\right\} + O(1). \tag{2}$$

If we define

$$Z_n(\theta, \theta + \frac{u}{\sqrt{n}}) = \log p(X^{(n)}; \theta + \frac{u}{\sqrt{n}}) - \log p(X^{(n)}; \theta)$$

$$Z_n(\theta, \theta + \frac{u}{\sqrt{n}}) = \int_\theta^{\theta + u/\sqrt{n}} E(\delta(\tilde{\beta})|X^{(n)})d\theta$$

From (2) we can deduce that

$$Z_n(\theta, \theta + \frac{u}{\sqrt{n}}) \xrightarrow{\mathcal{L}} u \int_0^1 \sqrt{2}\frac{\partial_\theta \sigma_s}{\sigma_s} dW_s - \frac{u^2}{2} \int_0^1 2 \left(\frac{\partial_\theta \sigma_s}{\sigma_s} \right)^2 ds,$$

that is the model satisfies the LAMN property. In particular this implies that the asymptotic Fisher information for θ is given by

$$2 \int_0^1 \left(\frac{\partial_\theta \sigma_s}{\sigma_s} \right)^2 ds.$$

Assume that our vector of observations $Z^{(n)}$ has parts in a Gaussian space ($\mathcal{G}$-measurable random variables $\mathcal{G}$ is the σ-field generated by $\mathcal{H}$, and parts "out" of a Gaussian space: independent of $\mathcal{G}$). For instance

$$Z_k = F(\theta, X_k, Y_k), k = 1, ..., n$$

where $X^{(n)} = (X_1, ..., X_n)$ is a $\mathcal{G}$-measurable random vector and $Y^{(n)} = (Y_1, ..., Y_n)$ is independent of $\mathcal{G}$. Let T be a regular statistics $T = T(Z^{(n)})$. Then we have

$$\begin{aligned} \partial_\theta E_\theta(T) &= \partial_\theta E(T(Z^{(n)})) = E(\partial_\theta T(Z^{(n)})) \\ &= E(\partial_{Z_i} T(Z^{(n)}) \partial_\theta Z_i). \end{aligned}$$

Now we can *define*

$$DZ_k := \partial_{X_k} F DX_k$$

where DX_k is the derivative when we are in a Gaussian space.

Also we can define the adjoint operator in the same way and with similar properties, in particular we have that if u is an element of $L^2(\Omega, \mathfrak{H})$ then we can write

$$E(\langle DT, u\rangle_{\mathfrak{H}}) = E(T\delta(u))$$

where δ is the adjoint operator of D and whenever u is in the domain of δ.

We have

$$DT = \partial_{Z_i} T(Z^{(n)}) DZ_i = \partial_{Z_i} T(Z^{(n)}) \partial_{X_i} F DX_i,$$

and by using the duality property, if we have a random variable V with values in $\mathfrak{H}$ (and in the domain of δ) such that

$$\langle DZ_i, V\rangle_{\mathfrak{H}} = \partial_\theta Z_i$$

then

$$\partial_\theta \log p(Z^{(n)}; \theta) = E(\delta(V)|Z^{(n)}).$$

Let $X^{(n)} = (X_{t_1}, X_{t_2}, ..., X_{t_n})$, $0 = t_0 < t_1 < t_2 ... < t_n$, where

$$X_t = x + \int_0^t b_s(\theta, X_s)ds + \int_0^t \sigma_s(\theta, X_s)dB_s + \int_0^t c_{s,z}(\theta, X_{s-})M(dz, ds).$$

Where $M(dx, ds)$ is a Poisson random measure with intensity $\nu(dz, ds) = \nu_s(dz)ds$ and B is an independent standard Brownian motion.

We again have

$$DX_t = \partial_x X_t \left(\partial_x X_\cdot\right)^{-1} \sigma_\cdot(\theta, X_\cdot)\mathbf{1}_{[0,t]}(\cdot).$$

Then ,

$$\partial_\theta \log p(x;\theta) = E(\delta(\tilde{\beta})|X^{(n)} = x),$$

where

$$\begin{aligned}\delta(\tilde{\beta}) &= \sum_{i=1}^{n}(\beta_{t_i} - \beta_{t_{i-1}}) \int_{t_{i-1}}^{t_i} a(t)\partial_x X_t \left(\sigma_t(\theta, X_t)\right)^{-1} dB_t \\ &\quad - \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} a(t) D_t \beta_{t_i} \partial_x X_t \left(\sigma_t(\theta, X_t)\right)^{-1} dt.\end{aligned}$$

If we take $\Delta t_i := t_i - t_{i-1} = 1/n$, and n observations, we have that

$$Z_n(\theta, \theta + \frac{u}{\sqrt{n}}) \xrightarrow{\mathcal{L}} u \int_0^1 \sqrt{2}\frac{\partial_\theta \sigma_s}{\sigma_s} dW_s - \frac{u^2}{2}\int_0^1 2\left(\frac{\partial_\theta \sigma_s}{\sigma_s}\right)^2 ds,$$

that is, the model satisfies the LAMN property too.

Let Y_t be an observation of a compound Poisson process of parameter λ and with jumps given by a random variable X. Then

$$\partial_\lambda E(\mathbf{1}_{\{Y_t\in A\}}) = \partial_\lambda \sum_{n=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^n}{n!} P(X_1 + ... + X_n \in A)$$

$$\begin{aligned}
&= -t\sum_{n=0}^{\infty}\left(\frac{e^{-\lambda t}(\lambda t)^n}{n!} - \frac{ne^{-\lambda t}(\lambda t)^{n-1}}{n!}\right) P(X_1 + ... + X_n \in A) \\
&= -tP(Y_t \in A) + \frac{1}{\lambda}\sum_{n=0}^{\infty}\frac{e^{-\lambda t}(\lambda t)^n}{n!} nP(X_1 + ... + X_n \in A) \\
&= -tE(\mathbf{1}_{\{Y_t\in A\}}) + \frac{1}{\lambda}E(\mathbf{1}_{\{Y_t\in A\}}N_t) = E(\mathbf{1}_{\{Y_t\in A\}}(\frac{N_t}{\lambda} - t)) \\
&= E(\mathbf{1}_{\{Y_t\in A\}}(\frac{N_T}{\lambda} - T)).
\end{aligned}$$

So the score function is given by

$$E(\frac{N_T}{\lambda} - T|Y_t)$$

Suppose now that our observations are $Y_{t_1}, .., Y_{t_n}, 0 < t_1 < .. < t_n \leq T$. Then, since the increments are independent, the score function is given by

$$E(\frac{N_T}{\lambda} - T|Y_{t_1}, Y_{t_2}, ..., Y_{t_n})$$

The trajectories of a compound process can be described by indicating the time and amplitude of the jumps. So $\Omega = \cup_{n=0}^{\infty} (\mathbb{R}_+ \times \mathbb{R})^n$ could serve for this purpose and if $\omega \in \cup_{n=0}^{\infty} (\mathbb{R}_+ \times \mathbb{R})^n$,
$\omega = ((s_1, x_1), ..., (s_k, x_k))$ for certain natural k. In this space we assume that $s_j - s_{j-1} \sim \exp(\lambda)$ $x_j \sim f(x)$ and all are independent. Then if we consider the random variables

$$\begin{aligned} Y_t : \Omega &\to \mathbb{R} \\ ((s_1, x_1), ..., (s_k, x_k)) &\to Y_t(\omega) = \sum_{i=1}^{k} x_i \mathbf{1}_{[s_i, \infty]}(t), \end{aligned}$$

$(Y_t)_{t \geq 0}$ is a compound Poisson process.

We have that

$$
\begin{aligned}
& \partial_\lambda E(\mathbf{1}_{\{Y_t \in A\}}) \\
= \; & t \sum_{n=0}^{\infty} \frac{e^{-\lambda t} (\lambda t)^n}{n!} (P(X_1 + .. + X_{n+1} \in A) - P(X_1 + .. + X_n \in A)) \\
= \; & tE(\sum_{n=0}^{\infty} \frac{e^{-\lambda t} (\lambda t)^n}{n!} (P(X_1 + . + X_n + x \in A) - P(X_1 + . + X_n \in A))_{x=X} \\
= \; & E \int_0^T E(\Psi_{s,x}(\mathbf{1}_{\{Y_t \in A\}}))_{x=X} ds = E \int_{\mathbb{R}} \int_0^T \Psi_{s,x}(\mathbf{1}_{\{Y_t \in A\}}) f(x) dx ds,
\end{aligned}
$$

where $\Psi_{s,x}(\mathbf{1}_{\{Y_t(\omega) \in A\}}) = \mathbf{1}_{\{Y_t(\omega_{s,x}) \in A\}} - \mathbf{1}_{\{Y_t(\omega) \in A\}}$, where $\omega_{s,x}$ is a trajectory like ω but where we add a jump with amplitude x at time s. We also can write

$$
\partial_\lambda E(\mathbf{1}_{\{Y_t \in A\}}) = \frac{1}{\lambda} E(\int_{\mathbb{R}} \int_0^T \Psi_{s,x}(\mathbf{1}_{\{Y_t \in A\}}) \nu(dx) ds),
$$

where $\nu(dx) = \lambda f(x) dx$ is the Lévy measure of the compound Poisson process Y.

If we denote by δ the adjoint operator of $\Psi_{\cdot,\cdot}$ we deduce from the previous calculations that the score function can be written as

$$E\left(\delta\left(\frac{1}{\lambda}\right)|Y_{t_1},..,Y_{t_n}\right)$$

Let $L_i := Y_{t_i} - Y_{t_{i-1}} = \int_{t_{i-1}}^{t_i} \int_{\mathbb{R}} g(\theta, x) M(dx, ds)$, $i = 1, ...n$, with

$$M(dx, ds) = \sum_{k=1}^{\infty} \delta_{\{\tau_k\}}(ds) \delta_{\{X_k\}}(dx),$$

where $0 < \tau_1 < \tau_2 < ... < \tau_n$ is a sequence of jump times of a Poisson process with intensity one and $X_{i,} i = 1, ..., n, ...$ is an iid sequence independent of $(\tau_i)_{i \geq 1}$ and with density f, $g(\theta, x)$ is a deterministic function. Note that there is not change in the intensity of the Lévy process but in the amplitude of the jumps.

In such a situation we have

$$\partial_\theta E(T) = \sum_{j=1}^{n} E(\partial_{Y_j} T \partial_\theta L_j),$$

and

$$\begin{aligned} \partial_\theta L_j &= \int_{t_{j-1}}^{t_j} \int_{\mathbb{R}} \partial_\theta g(\theta, x) M(dx, ds) \\ &= \sum_{k=1}^{\infty} \partial_\theta g(\theta, X_k) \mathbf{1}_{(\tau_k \in (t_{j-1}, t_j)\}}, \end{aligned}$$

Then we can write

$$
\begin{aligned}
\partial_\theta E(T(Y)) &= \sum_{j=1}^{n}\sum_{k=1}^{\infty} E(\partial_{L_j} T \partial_{X_k} L_j \partial_\theta g(\theta, X_k) \frac{1}{\partial_x g(\theta, X_k)} \mathbf{1}_{(\tau_k \in (t_{j-1}, t_j)\}}) \\
&= \sum_{j=1}^{n}\sum_{k=1}^{\infty} E(\partial_{X_k} T \frac{\partial_\theta g(\theta, X_k)}{\partial_x g(\theta, X_k)} \mathbf{1}_{(\tau_k \in (t_{j-1}, t_j)\}}) \\
&= -\sum_{j=1}^{n}\sum_{k=1}^{\infty} E\left(T \frac{1}{f(X_k)} \partial_x \left(\frac{\partial_\theta g(\theta, x) f(x)}{\partial_x g(\theta, x)} \right)_{x=X_k} \mathbf{1}_{(\tau_k \in (t_{j-1}, t_j)\}} \right),
\end{aligned}
$$

where in the last equality we use the independence between (X_k) and the jump times and where we assume there are not "border" effects .

Finally

$$\begin{aligned}
\partial_\theta E(T) &= -\sum_{j=1}^{n}\sum_{k=1}^{\infty} E\left(T\frac{1}{f(X_k)}\partial_x\left(\frac{\partial_\theta g(\theta,x)f(x)}{\partial_x g(\theta,x)}\right)_{x=X_k}\mathbf{1}_{(\tau_k\in(t_{j-1},t_j)\}}\right) \\
&= -E\left(T\sum_{j=1}^{n}\int_{\mathbb{R}}\int_{t_{j-1}}^{t_j}\frac{1}{f(x)}\partial_x\left(\frac{\partial_\theta g(\theta,x)f(x)}{\partial_x g(\theta,x)}\right)M(dx,ds)\right) \\
&= -E\left(T\int_{\mathbb{R}}\int_{t_0}^{t_n}\frac{1}{f(x)}\partial_x\left(\frac{\partial_\theta g(\theta,x)f(x)}{\partial_x g(\theta,x)}\right)M(dx,ds)\right).
\end{aligned}$$

So, the score function is given by

$$-E(\int_{\mathbb{R}}\int_{t_0}^{t_n}\frac{1}{f(x)}\partial_x\left(\frac{\partial_\theta g(\theta,x)f(x)}{\partial_x g(\theta,x)}\right)M(dx,ds)|Y_1,Y_2,\ldots,Y_n).$$

CLT. Random variables in a fixed chaos

Theorem

Fix $n \geq 2$. Consider a sequence $\{F_k = I_n(f_k), k \geq 1\}$ such that

$$E(F_k^2) \underset{k\to\infty}{\to} \sigma^2 \tag{3}$$

The following statements are equivalent:

(i) $F_k \overset{\mathcal{L}}{\underset{k\to\infty}{\to}} N(0,\sigma^2)$.

(ii) $E(F_k^4) \underset{k\to\infty}{\to} 3\sigma^4$.

(iii) $\|f_k \otimes_r f_k\|_{\mathfrak{H}^{\otimes 2(n-r)}} \underset{k\to\infty}{\to} 0$, *for all* $1 \leq r \leq n-1$.

(iv) $\|DF_k\|_{\mathfrak{H}}^2 \overset{L^2(\Omega)}{\underset{k\to\infty}{\to}} n\sigma^2$.

CLT. Random variables in a fixed chaos

Example

Consider a sequence of stationary, normalized, centered Gaussian random variables $(X_i)_{i\geq 1}$. We want to study the asymptotic behavior of the sequence

$$F_k := \frac{1}{\sqrt{k}} \sum_{i=1}^{k} H_m(X_i) \ ,$$

$m \geq 2$. We can take $\mathcal{H}_1 = span\{X_i, i \geq 1\}$, and $\mathfrak{H} \equiv \mathcal{H}_1$. The inner product on $\mathfrak{H}$ is then induced by the covariance function $\rho(k) = \mathrm{cov}(X_1, X_{1+k})$ of the sequence $(X_i)_{i\geq 1}$ (note that $\rho(0) = 1$). We obtain the following representation

$$F_k = \frac{1}{\sqrt{k}} \sum_{i=1}^{k} H_m(X_i) = I_m\left(\frac{1}{\sqrt{k}} \sum_{i=1}^{k} X_i^{\otimes m}\right) .$$

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CLT. Random vectors with components in fixed chaos

For $d \geq 2$, fix d natural numbers, $1 \leq n_1 \leq ... \leq n_d$. Consider a sequence of random vectors

$$F_k = (F_k^1, .., F_k^d) = (I_{n_1}(f_k^1), ..., I_{n_d}(f_k^d)), \qquad (4)$$

where $f_k^i \in \mathfrak{H}^{\odot n_i}$. We have a multidimensional version of the previous theorem,

CLT. Random vectors with components in fixed chaos

Theorem

Let $(F_k)_{k\geq 1}$ be a sequence of random vectors of the form (4) such that, for every $1 \leq i, j \leq d$

$$\lim E(F_k^i F_k^j) = \delta_{ij}, \tag{5}$$

then the following statements are equivalent

(i) For every $i = 1, ..., d,\ F_k^i \underset{k\to\infty}{\overset{\mathcal{L}}{\to}} N(0,1).$

(ii) For every $i = 1, ..., d,\ E(\left(F_k^i\right)^4) \underset{k\to\infty}{\to} 3.$

(iii) $\left\| f_k^i \otimes_r f_k^i \right\|_{\mathfrak{H}^{\otimes 2(n_i - r)}} \underset{k\to\infty}{\to} 0,$ *for all* $1 \leq r \leq n_i - 1, 1 \leq i \leq d$.

(iv) For every $i = 1, ..., d, \left\| DF_k^i \right\|_{\mathfrak{H}}^2 \underset{k\to\infty}{\overset{L^2(\Omega)}{\to}} n_i.$

(v) $F_k \underset{k\to\infty}{\overset{\mathcal{L}}{\to}} N_d(0, I_d)$

Finally, we can consider a d-dimensional random vector $F_k = (Y_k^1, \ldots, Y_k^d)^T$ which has a chaos representation

$$F_k^i = \sum_{m=1}^{\infty} I_m(f_{m,k}^i) \,, \qquad i = 1, \ldots, d \,,$$

with $f_{m,k}^i \in \mathfrak{H}^{\odot m}$.

Theorem

Suppose that the following conditions hold:

(i) *For any* $i = 1, \ldots, d$ *we have* $\sum_{m=1}^{\infty} \sup_k m! ||f^i_{m,k}||^2_{\mathfrak{H}^{\otimes m}} < \infty$.

(ii) *For any* $m \geq 1$, $i, j = 1, \ldots, d$ *we have constants* Σ^m_{ij} *such that*

$$\lim_{k\to\infty} E[I_m(f^i_{m,k}) I_m(f^j_{m,k})] = \lim_{k\to\infty} \left\langle f^i_{m,k}, f^j_{m,k} \right\rangle_{\mathfrak{H}^{\odot m}} = \Sigma^m_{ij},$$

and the matrix $\Sigma^m = (\Sigma^m_{ij})_{1\leq i,j\leq d}$ *is positive definite for all m.*

(iii) $\sum_{m=1}^{\infty} \Sigma^m = \Sigma \in \mathbb{R}^{d\times d}$.

(iv) *For any* $m \geq 1$, $i = 1, \ldots, d$ *and* $p = 1, \ldots, m-1$

$$\lim_{k\to\infty} ||f^i_{m,k} \otimes_p f^i_{m,k}||^2_{\mathfrak{H}^{\otimes 2(m-p)}} = 0.$$

Then we have $F_k \xrightarrow[k\to\infty]{\mathcal{L}} N_d(0, \Sigma)$.

Let $(G_t)_{t\geq 0}$ be a Gaussian process which has centered and stationary increments. We want to study the asymptotic properties of the process

$$V(G,p)_t^n = \frac{1}{n\tau_n^p}\sum_{i=1}^{[nt]} |\Delta_i^n G|^p \ ,$$

where $\Delta_i^n G = G_{\frac{i}{n}} - G_{\frac{i-1}{n}}$, $\tau_n^2 = E[|\Delta_i^n G|^2]$ and $p > 0$. Write

$$r_n(j) = \mathrm{Cov}\Big(\frac{\Delta_1^n G}{\tau_n}, \frac{\Delta_{1+j}^n G}{\tau_n}\Big) \ , \qquad j \geq 0.$$

and assume that

$$|r_n(j)|^2 \leq Cj^{-1-\varepsilon},\ j \geq 0, \text{for some } \varepsilon > 0 \tag{6}$$

and

$$\lim_{n\to\infty} r_n(j) = \rho(j),$$

Set $H(x) = |x|^p - \mu_p$, where $\mu_p = E(|N(0,1)|^p)$, then $H(x) = \sum_{j=2}^{\infty} a_j H_j(x)$, with $a_2 > 0$ and we have the following theorem:

Theorem

$$\left(G_t, \sqrt{n}(V(G,p)_t^n - t\mu_p)\right) \underset{n\to\infty}{\xrightarrow{\mathcal{L}}} \left(G_t, \sigma W_t\right),$$

where W is a Brownian motion that is defined on an extension of the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$, independent of G and σ^2 is given by

$$\sigma^2 = \sum_{m=2}^{\infty} \sigma_m^2, \quad \sigma_m^2 = m! a_m^2 \lambda_m^2, \quad \lambda_m^2 = 1 + 2\sum_{i=1}^{\infty} \rho^m(i). \tag{7}$$

Let $\mathcal{H}_1$ the closed subspace of $L^2(\Omega, \mathcal{F}, P)$ generated by the random variables $(\Delta_j^n G/\tau_n)_{n\geq 1, 1\leq j\leq [nT]}$. Notice that $\mathcal{H}_1$ is a separable Hilbert space with the scalar product induced by the covariance function of the triangular array $(\Delta_j^n G/\tau_n)_{n\geq 1, 1\leq j\leq [nT]}$. Then we can take $\mathfrak{H} = \mathcal{H}_1$ and try to apply the general CLT to this case.

We can also study with similar tools the multipower variation

$$V(G, p_1, \ldots, p_k)_t^n = \frac{1}{n} \sum_{i=1}^{[nt]-k+1} \prod_{j=1}^{k} \left| \frac{\Delta_{i+j-1}^n G}{\tau_n} \right|^{p_j} , \quad p_1, \ldots, p_k \geq 0 ,$$

and for the **joint** multipower variation:

$$\left(V(G, p_1^1, \ldots, p_k^1)_t^n, ..., V(G, p_1^d, \ldots, p_k^d)_t^n \right) .$$

Define

$$\rho^{(n)}_{p_1,\dots,p_k} = E\left[\left|\frac{\Delta^n_1 G}{\tau_n}\right|^{p_1} \cdots \left|\frac{\Delta^n_k G}{\tau_n}\right|^{p_k}\right].$$

We have

Theorem

$$\left(G_t, \sqrt{n}\left(V(p^j_1,\dots,p^j_k)^n_t - \rho^{(n)}_{p^j_1,\dots,p^j_k}\, t\right)_{1\le j\le d}\right) \to (G_t, \beta^{1/2} W_t),$$

where W is a d-dimensional Brownian, defined in an extension of the original filtered space, independent of G, β is a $d\times d$-dimensional matrix given by

$$\beta_{ij} = \lim_{n\to\infty} n\, cov\left(V_Q(p^i_1,\dots,p^i_k)^n_1, V_Q(p^j_1,\dots,p^j_k)^n_1\right), \quad 1 \le i,j \le d,$$

and $(Q_i)_{i\ge 1}$ is stationary centered discrete time Gaussian process with correlation function $\rho(j)$.

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