Some appllications of the Lent Particle Method

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↪ *Based on joint works with N. Bouleau.*

references

- Bouleau N. and Denis L. "Energy image density property and the lent particle method for Poisson measures" *Jour. of Functional Analysis* 257 (2009) 1144-1174.
- Bouleau N. and Denis L. "Application of the lent particle method to Poisson driven SDE's", to appear in Probability Theory and Related Fields.
- Bouleau N. and Denis L. " Application of the lent particle method to existence of smooth densities of Poisson functionals", submitted.

Introduction

We are given

• N: a Poisson random measure on $[0,+\infty[\times\Xi$ with intensity $dt\times\nu(du)$ defined on the probability space $(\Omega_1,\mathcal{A}_1,\mathbb{P}_1)$ where Ω_1 is the **configuration space**.

• $(\Xi,\mathcal{X},\nu,\mathbf{d},\gamma)$: a local symmetric Dirichlet structure which admits a carré du champ operator i.e. $(\Xi,\mathcal{X},\nu)$ is a measured space, ν is σ-finite and the bilinear form

$$e[f,g]=\frac{1}{2}\int\gamma[f,g]\,d\nu,$$

is a local Dirichlet form with domain $\mathbf{d}\subset L^2(\nu)$.

• $u\mapsto u^\flat$, a version of the gradient on $\mathbf{d}$ with values in the space $L^2_0(R,\mathcal{R},\rho)=\{g\in L^2(R,\mathcal{R},\rho);\int_R g(r)\rho(dr)=0\}$.

Example: the finite dimensional case

Let $(\Xi, \mathcal{X}) = (\mathbb{R}^r, \mathcal{B}(\mathbb{R}^r))$ and $\nu = k dx$ where $k \geq 0$.
We assume that there exist an open set $O \subset \mathbb{R}^r$ and a function ψ in $H^1_{loc}(O)$ continuous on O and null on $\mathbb{R}^r \setminus O$ such that

1. $k > 0$ on O ν-a.e. and is locally bounded on O
2. $k \geq \psi > 0$ ν-a.e. on O.

We denote by H the subspace of functions $f \in L^2(\nu) \cap L^1(\nu)$ such that the restriction of f to O belongs to $C_c^\infty(O)$. Then, the bilinear form defined by

$$\forall f, g \in H,\ e(f,g) = \sum_{i=1}^r \int_O |x|^2 \partial_i f(x) \partial_i g(x) \psi(x)\, dx$$

is closable in $L^2(\nu)$. Its closure, $(\mathbf{d}, e)$, is a local Dirichlet form on $L^2(\nu)$ which admits a carré du champ γ.

$$\forall f \in \mathbf{d},\ \gamma(f)(x) = \sum_{i=1}^r |x|^2 |\partial_i f(x)|^2 \frac{\psi(x)}{k(x)}.$$

Creation, annihilation operators and

$$\varepsilon^+_{(t,u)}(w_1) = w_1 \mathbf{1}_{\{(t,u)\in supp\, w_1\}} + (w_1 + \varepsilon_{(t,u)}\})\mathbf{1}_{\{(t,u)\notin supp\, w_1\}}$$
$$\varepsilon^-_{(t,u)}(w_1) = w_1 \mathbf{1}_{\{(t,u)\notin supp\, w_1\}} + (w_1 - \varepsilon_{(t,u)}\})\mathbf{1}_{\{(t,u)\in supp\, w_1\}}.$$

In a natural way, we extend these operators to the functionals by

$$\varepsilon^+ H(w_1, t, u) = H(\varepsilon^+_{(t,u)} w_1, t, u) \quad \varepsilon^- H(w_1, t, u) = H(\varepsilon^-_{(t,u)} w_1, t, u).$$

• $N \odot \rho$ the *extended marked Poisson measure*: it is a random Poisson measure on $[0, +\infty[\times \Xi \times R$ with compensator $dt \times \nu \times \rho$ defined on the product probability space: $(\Omega_1, \mathcal{A}_1, \mathbb{P}_1) \times (\hat{\Omega}, \hat{\mathcal{A}}, \hat{\mathbb{P}})$, where $(\hat{\Omega}, \hat{\mathcal{A}}, \hat{\mathbb{P}}) = (R^{\mathbb{N}}, \mathcal{R}^{\otimes\mathbb{N}}, P^{\otimes\mathbb{N}})$.

Upper Dirichlet structure

From this, we are able to construct a Dirichlet form $(\mathbb{D}, \mathcal{E}, \Gamma)$ on $L^2(\Omega)$ which admits a carré du champ Γ and a gradient operator that we denote by $\sharp$ and given by the following formula:

$$\forall F \in \mathbb{D}, \quad F^\sharp = \int_0^{+\infty} \int_{\Xi \times R} \varepsilon^-((\varepsilon^+ F)^\flat) \, dN \odot \rho \in L^2(\mathbb{P}_1 \times \hat{\mathbb{P}}). \quad (1)$$

Moreover, we have

- for all $F \in \mathbb{D}$, $\Gamma[F] = \hat{\mathbb{E}}[|F^\sharp|^2] = \int_0^{+\infty} \int_\Xi \varepsilon^-(\gamma[\varepsilon^+ F]) \, dN$.
- If $h \in L^2(\mathbb{R}^+, dt) \otimes \mathbf{d}$, then $\tilde{N}(h) = \int_0^{+\infty} \int_\Xi h(t,u) \tilde{N}(ds, du)$ belongs to $\mathbb{D}$ and

$$\Gamma[\tilde{N}(h)] = \int_0^{+\infty} \int_\Xi \gamma[h(t,\cdot)](u) N(dt, du) = N(\gamma[h]).$$

$$\left(\tilde{N}(h)\right)^\sharp = \int_0^{+\infty} \int_{\Xi \times R} h^\flat(t,u,r) N \odot \rho(dt, du, dr).$$

Definition and hypotheses on the bottom space

Let E be a Hilbert space. We denote by $\mathbf{d}(E)$ the completion of functions of the form

$$u = \sum_{i=1}^{k} \varphi_i e_i$$

with $e_1, \cdots, e_k$ in E and $\varphi_1, \cdots, \varphi_k$ in $\mathbf{d}$ w.r.t. the norm

$$\| u\|^2_{\mathbf{d}(E)} = \|u\|^2_{L^2(\nu;E)} + \|u^\flat\|^2_{L^2(\nu;L_0\otimes E)}. \qquad (2)$$

Here $u^\flat(r) = \sum_{i=1}^k \varphi_i^\flat e_i$.

Hypothesis (C): There exists a dense subvector space $\mathbf{d}_0 \subset \mathbf{d}$ such that each element u in $\mathbf{d}_0$ is such that:

1. $u \in \bigcap_{p\geq 2} L^p(\nu)$.
2. u is *infinitely differentiable* in the sense that $u^\flat \in \mathbf{d}(L_0)$, $u^{(2\flat)} = \left(u^\flat\right)^\flat \in \mathbf{d}(L_0^{\otimes 2}), \dots, u^{((n+1)\flat)} = \left(u^{(n\flat)}\right)^\flat \in \mathbf{d}(L_0^{\otimes (n+1)}) \dots$
3. For all $n \in \mathbb{N}^*$, $u^{(n\flat)} \in \bigcap_{p\geq 2} L^p(\nu; L_0^{\otimes n})$.

$$\mathbf{d}_0(E) = \{u = \sum_{i=1}^{n} \varphi_i e_i |\, \varphi_i \in \mathbf{d}_0,\ i = 1, \cdots, n\}.$$

Definition

Let $n \in \mathbb{N}^*$, $p \geq 2$. We denote by $\mathbf{d}^{n,p}(E)$ the completion of $\mathbf{d}_0(E)$ w.r.t. the norm

$$\|u\|_{n,p} = \|u\|_{L^p(\nu;E)} + \|u^\flat\|_{L^p(\nu;L_0\otimes E)}\| + \cdots + \|u^{(n\flat)}\|_{L^p(\nu;L_0^{\otimes n}\otimes E)}.$$

And we set:

$$\mathbf{d}^\infty(E) = \bigcap_{n\in\mathbb{N}^*, p\geq 2} \mathbf{d}^{n,p}(E).$$

Definition

We denote by $\bar{\mathbf{d}}^\infty$ *the subvector space of elements* u *in* $\mathbf{d}^\infty$ *such that* u *belongs to* $\mathcal{D}(a)$ *and* $a(u) \in \mathbf{d}^\infty$ *where* a *is the generator of* $(\mathbf{d}, e)$*. Let* $n \in \mathbb{N}^*$*,* $p \geq 2$*. We denote by* $\bar{\mathbf{d}}^{n,p}$ *the completion of* $\bar{\mathbf{d}}_0$ *w.r.t. the norm*

$$\|u\|_{\bar{\mathbf{d}}^{n,p}} = \|u\|_{n,p} + \|a(u)\|_{n,p}$$

And we set:

$$\bar{\mathbf{d}}^\infty = \bigcap_{n\in\mathbb{N}^*, p\geq 2} \bar{\mathbf{d}}^{n,p}.$$

Remark

It is necessary to introduce these spaces $\bar{\mathbf{d}}^{n,p}$ since, in the general case:

$$u \in \mathbf{d}^{2,2} \not\Rightarrow u \in \mathcal{D}(a)$$

in other words:

$$\bar{\mathbf{d}}^{n,p} \underset{\neq}{\subseteq} \mathbf{d}^{n,p}$$

$\hookrightarrow$ Some Meyer-type inequalities are required to have equality.

Sobolev spaces on the upper space

We follow the same construction as on the bottom space, starting from

$$\mathbb{D}_0 = \left\{ \varphi(\tilde{N}(f_1), \cdots, \tilde{N}(f_k)) |\ k \in \mathbb{N}^*, \varphi \in C_c^\infty(\mathbb{R}^k), f_i \in \mathbf{d}_\infty\ i = 1, \cdots, k \right\}$$

$$\mathbb{D}_0(E) = \{ \sum_{i=1}^k G_i e_i |\ k \in \mathbb{N}^*, G_i \in \mathbb{D}_0,\ e_i \in E\ \ i = 1, \cdots, k \}.$$

$X^{(n\sharp)}$: the derivate of $X^{((n-1)\sharp)} \in \mathbb{D}\left(L^2(\hat{\mathbb{P}}^{(n-1)}; E) \right)$ so it belongs to $\mathbb{D}\left(L^2(\hat{\mathbb{P}}^n; E) \right) \subset L^2(\mathbb{P} \times \hat{\mathbb{P}}^n; E)$.

Definition

Let $n \in \mathbb{N}^*, p \geq 2$ the Sobolev space $\mathbb{D}^{n,p}(E)$ is the closure of $\mathbb{D}_0(E)$ with respect to the norm

$$\|X\|_{n,p} = \|X\|_{L^p(\mathbb{P};E)} + \|X^\sharp\|_{L^p(\mathbb{P}\times\hat{\mathbb{P}};E)}\| + \cdots + \|X^{(n\sharp)}\|_{L^p(\mathbb{P}\times\hat{\mathbb{P}}^n;E)},$$

and $\mathbb{D}^\infty(E) = \bigcap_{n\in\mathbb{N}^*, p\geq 2} \mathbb{D}^{n,p}(E)$.

In the same way as in the previous subsection, for all $n \in \mathbb{N}^*, p \geq 2$ we consider first $\bar{\mathbb{D}}^{\infty}$, the subvector space of elements in $\mathbb{D}^{\infty} \bigcap \mathcal{D}(A)$ such that $A(X) \in \mathbb{D}^{\infty}$, where A is the generator of $(\mathbb{D}, \mathcal{E})$.
We consider the space $\bar{\mathbb{D}}^{n,p}$ which is the closure of $\bar{\mathbb{D}}_0$ with repect to the norm

$$\|X\|_{\bar{\mathbb{D}}^{n,p}} = \|X\|_{n,p} + \|A(X)\|_{n,p},$$

and put

$$\bar{\mathbb{D}}^{\infty} = \bigcap_{n \in \mathbb{N}^*, p \geq 2} \bar{\mathbb{D}}^{n,p}.$$

Representation of the n-order derivative

For all $n \in \mathbb{N}^*$, we construct a random Poisson measure $N \odot \rho^n$ on $[0,+\infty[\times X \times R^n$ with compensator $dt \times \nu \times \underbrace{\rho \times \cdots \times \rho}_{n\ times}$ defined on the product probability space: $(\Omega_1, \mathcal{A}_1, \mathbb{P}_1) \times (\hat{\Omega}, \hat{\mathcal{A}}, \hat{\mathbb{P}})^{\mathbb{N}^*}$.

Lemma

Let $h \in L^2(\mathbb{R}^+, dt) \otimes \mathbf{d}^\infty$, then $\tilde{N}(h) = \int_0^{+\infty} \int_X h(t,u) \tilde{N}(ds, du)$ belongs to $\mathbb{D}^\infty$ and for all $n \in \mathbb{N}^*$:

$$\tilde{N}(h)^{(n\sharp)} = \int_0^{+\infty} \int_{X \times R^n} h^{(n\flat)}(t,u,r_1,\cdots,r_n) N \odot \rho^n(dt, du, dr_1, \cdots, r_n).$$

Properties of these Sobolev spaces

- $X \in \mathbb{D}^\infty$, $Y \in \mathbb{D}^\infty(E) \Rightarrow XY \in \mathbb{D}^\infty(E)$
- $X \in \mathbb{D}^\infty(E) \Rightarrow Y = \|X\|_E^2 \in \mathbb{D}^\infty$.
- $X \in \mathbb{D}^\infty \Rightarrow \Gamma[X] \in \mathbb{D}^\infty$.
- Let $X \in \mathbb{D}^\infty$ be positive and such that $\frac{1}{X} \in \bigcap_{p \geq 2} L^p(\mathbb{P})$, then

$$\frac{1}{X} \in \mathbb{D}^\infty.$$

The main result

The operator $X \mapsto X^\sharp$, considered as an unbounded operator with domain $\mathbb{D} \subset L^2(\mathbb{P})$ and values in $L^2(\mathbb{P} \times \hat{\mathbb{P}})$, admits an adjoint operator that we denote by $\delta_\sharp : \mathcal{D}(\delta_\sharp) \subset L^2(\mathbb{P} \times \hat{\mathbb{P}}) \to L^2(\mathbb{P})$.

Lemma

Let $X \in \mathbb{D}^\infty$ and $Y \in \bar{\mathbb{D}}^\infty$ then $XY^\sharp$ belongs to $\mathcal{D}(\delta_\sharp)$ and

$$\delta_\sharp[XY^\sharp] = -2XAY - \Gamma[X, Y].$$

This yields easily:

Proposition

Let $d \in \mathbb{N}^*$ and X be in $\left(\bar{\mathbb{D}}^\infty\right)^d$. If $(\Gamma[X])^{-1} \in \bigcap_{p \geq 2} L^p(\mathbb{P}; \mathbb{R}^{d \times d})$, then X admits a density which belongs to $C_b^\infty(\mathbb{R}^d)$.

Meyer Inequalities

First of all, we recall that thanks to Khintchine's inequalities:

Proposition

By choosing well the probability space $(R, \mathcal{R}, \rho)$ and the version of the gradient on the bottom space then the norm on $\mathbb{D}^{k,p}$ is equivalent to the following norm

$$\|F\|_{L^p(\mathbb{P})} + \sum_{i=1}^{k} \|(\Gamma_i[F])^{1/2}\|_{L^p(\mathbb{P})},$$

where $\Gamma_i[F] = \hat{\mathbb{E}}[(F^{(i\sharp)})^2]$.

Some examples for which Meyer inequalities hold

Proposition

If
(i) $\Xi = \mathbb{R}^d$, ν *is the Lebesgue measure and* $a = \frac{1}{2}\Delta$*;*
or
(ii) the bottom space is an abstract Wiener space equipped with the Ornstein-Uhlenbeck structure;
then

$$c_p \|(-A)^{k/2} F\|_p \leq \|\sqrt{\Gamma_k[F]}\|_p \leq C_p \|(-A)^{k/2} F\|_p. \qquad (3)$$

Remark: In case *(i)* this result was proved in 1987 by L. Wu.

The SDE we consider

We consider another probability space $(\Omega_2, \mathcal{A}_2, \mathbb{P}_2)$ on which an $\mathbb{R}^n$-valued semimartingale $Z = (Z^1, \cdots, Z^n)$ is defined, $n \in \mathbb{N}^*$.
Assumption on Z: There exists a positive constant C such that for any square integrable $\mathbb{R}^n$-valued predictable process h:

$$\forall t \geq 0,\ \mathbb{E}[(\int_0^t h_s dZ_s)^2] \leq C^2 \mathbb{E}[\int_0^t |h_s|^2 ds]. \tag{4}$$

We shall work on the product probability space:
$(\Omega, \mathcal{A}, \mathbb{P}) = (\Omega_1 \times \Omega_2, \mathcal{A}_1 \otimes \mathcal{A}_2, \mathbb{P}_1 \times \mathbb{P}_2)$.
Let $d \in \mathbb{N}^*$, we consider the following SDE :

$$X_t = x + \int_0^t \int_\Xi c(s, X_{s^-}, u) \tilde{N}(ds, du) + \int_0^t \sigma(s, X_{s^-}) dZ_s \tag{5}$$

where $x \in \mathbb{R}^d$, $c : \mathbb{R}^+ \times \mathbb{R}^d \times \Xi \rightarrow \mathbb{R}^d$ and $\sigma : \mathbb{R}^+ \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times n}$ satisfy the set of hypotheses below denoted $(\bar{\mathrm{R}})$.

Hypotheses ($\bar{\mathrm{R}}$)

1.a) For $\mathbb{P}_2$-almost all $w_2 \in \Omega_2$, all $t \in [0,T]$ and $u \in \Xi$, $c(t,\cdot,u)$ is infinitely differentiable and

$$\forall \alpha \in \mathbb{N}^*, \quad \sup_{t\in[0,T],x\in\mathbb{R}^d} |D_x^\alpha c(t,x,\cdot)| \in \bigcap_{p\geq 1} L^p(\Omega_2 \times \Xi, \mathbb{P}_2 \times \nu),$$

b) $\sup_{t\in[0,T]} |c(t,0,\cdot)| \in \bigcap_{p\geq 1} L^p(\Omega_2 \times \Xi, \mathbb{P}_2 \times \nu)$,
c) for all $t \in [0,T]$, $\alpha \in \mathbb{N}$ and $x \in \mathbb{R}^d$, $D_x^\alpha c(t,x,\cdot) \in \bar{\mathbf{d}}^\infty$ and

$$\forall n \in \mathbb{N}^*, \ \forall p \geq 1, \quad \sup_{t\in[0,T],x\in\mathbb{R}^d} \|D_x^\alpha c(t,x,\cdot)\|_{\bar{\mathbf{d}}^{n,p}} < +\infty.$$

d) for all $t \in [0,T]$, all $x \in \mathbb{R}^d$ and $u \in \Xi$, the matrix $I + D_x c(t,x,u)$ is invertible and

$$\sup_{t\in[0,T],x\in\mathbb{R}^d} \left| (I + D_x c(t,x,u))^{-1} \times c(t,x,u) \right| \in \bigcap_{p\geq 1} L^p(\Omega_2 \times \Xi, \mathbb{P}_2 \times \nu).$$

2. For all $t \in [0, T]$, $\sigma(t, \cdot)$ is infinitely differentiable and

$$\forall \alpha \in \mathbb{N}^* \sup_{t\in[0,T],x\in\mathbb{R}^d} |D_x^\alpha \sigma(t,x)| \in \bigcap_{p\geq 1} L^p(\Omega_2, \mathbb{P}_2).$$

3. As a consequence of hypotheses 1. and 2. above, it is well known that equation (5) admits a unique solution X such that $\mathbb{E}[\sup_{t\in[0,T]} |X_t|^2] < +\infty$. We suppose that for all $t \in [0, T]$, the matrix $(I + \sum_{j=1}^n D_x\sigma_{\cdot,j}(t, X_{t^-})\Delta Z_t^j)$ is invertible and its inverse is bounded by a deterministic constant uniformly with respect to $t \in [0, T]$.

Functional calculus

Consider for all $t \in [0, T]$ $X_t = \int_0^t \int_\Xi H(s,u)\tilde{N}(ds,du)$.
We have:

$$\begin{aligned} X_t^\sharp(w,w_1) &= \int_0^t \int_\Xi H^\sharp(s,u)(w,w_1)\tilde{N}(ds,du)(w) \\ &+ \int_0^t \int_{\Xi\times R} H^\flat(s,u,r_1)(w) N\odot\rho(ds,du,dr_1)(w,w_1), \end{aligned}$$

$$\begin{aligned} X_t^{(2\sharp)}(w,w_1,w_2) &= \int_0^t \int_\Xi H^{2\sharp}(s,u)(w,w_1,w_2)\tilde{N}(ds,du)(w) \\ &+ \int_0^t \int_{\Xi\times R} H^{\sharp,\flat}(s,u,r_1)(w,w_1) N\odot\rho(ds,du,dr_1)(w,w_2) \\ &+ \int_0^t \int_{\Xi\times R} H^{\sharp,\flat}(s,u,r_1)(w,w_2) N\odot\rho(ds,du,dr_1)(w,w_1) \\ &+ \int_0^t \int_{\Xi\times R^2} H^{(2\flat)}(s,u,r_1,r_2)(w) N\odot\rho^{\odot 2}(ds,du,dr_1,dr_2)(w,w_1,w_2). \end{aligned}$$

More generally, for all $n \in \mathbb{N}^*$,

$$X_t^{(n\sharp)} = \sum_{i=1}^{2^n} I_i,$$

where $I_1 = \int_0^t \int_\Xi H^{(n\sharp)}(s,u)\tilde{N}(ds,du)$ and for $i \in \{2, \cdots, 2^n\}$, I_i is a term of the form

$$I_i(w, w_1, \cdots, w_n) =$$
$$\int_0^t \int_{\Xi \times R^j} H^{(j\sharp),(n-j)\flat}_{(s,u,r_1,\cdots,r_{n-j})}(w, w_{\sigma(1)}, \cdots, w_{\sigma(j)}) N \odot \rho^{\odot n-j},$$

where $j \in \{0, \cdots, n-1\}$ and σ is a permutation on $\{1, \cdots, n\}$.

How does the generator operate on stochastic integrals?

Proposition

Consider

$$X_t = \int_0^t \int_X H(s,u)\tilde{N}(ds,du)$$

then

$$A[X_t] = \int_0^t \int_X \left(A[H(s,u)] + a[H(s,\cdot)](u)\right) \tilde{N}(ds,du).$$

The case of integrals w.r.t. Z

Proposition

Consider

$$X_t = \int_0^t \int_X G_s \, dZ_s$$

then

$$X_t^{(n\sharp)} \quad = \quad \int_0^t G_s^{(n\sharp)} \, dZ_s.$$

Moreover,

$$A[X_t] = \int_0^t \int_X A[G_s] \, dZ_s.$$

Spaces of processes

- $\mathcal{H}_{\bar{\mathbb{D}}^{n,p},\mathcal{P}}$: the set of predictable real valued processes which belong to $L^2([0,T];\bar{\mathbb{D}}^{n,p})$.

In a natural way, we set

$$\mathcal{H}_{\bar{\mathbb{D}}^{\infty},\mathcal{P}} = \bigcap_{n\in\mathbb{N}^*,p\geq 2} \mathcal{H}_{\bar{\mathbb{D}}^{n,p},\mathcal{P}}, \quad .$$

Regularity of the solution

Applying the Picard iteration, we get:

Proposition

Under hypotheses $(\bar{R})$, *the equation* (5) *admits a unique solution,* X, *in* $(\mathcal{H}_{\bar{\mathbb{D}}^\infty,\mathcal{P}})^d$ *and we have for all* $t \in [0,T]$ *and all* $i \in \{1, \cdots, d\}$*:*

$$A[X_{i,t}] = \int_0^t \int_X a[c_i(s, X_{s^-}, \cdot)](u)\, \tilde{N}(ds, du) +$$

$$\int_0^t \int_X \left(\frac{\partial c_i}{\partial x_j}(s, X_{s^-}, u) A[X_{j,s^-}] + \frac{1}{2} \frac{\partial^2 c_i}{\partial x_j \partial x_k}(s, X_{s^-}, u) \Gamma[X_{j,s^-}, X_{k,s^-}] \right) \tilde{N}(ds, du$$

$$+ \int_0^t \left(\frac{\partial \sigma_i}{\partial x_j}(s, X_{s^-}) A[X_{j,s^-}] + \frac{1}{2} \frac{\partial^2 \sigma_i}{\partial x_j \partial x_k}(s, X_{s^-}) \Gamma[X_{j,s^-}, X_{k,s^-}] \right) dZ_s$$

Obtaining the carré du champ matrix

By applying the lent particle method, we get easily:

Theorem

For all $t \in [0, T]$,

$$\Gamma[X_t] \quad = \quad K_t \int_0^t \int_\Xi \bar{K}_s \gamma[c(s, X_{s-}, \cdot)] \bar{K}_s^* \, N(ds, du) K_t^*,$$

where the process (K_t) is the derivate of the flow and $(\bar{K}_t)$ its inverse.

Application: the regular case

Proposition

Assume hypotheses $(\bar{R})$, that ν has an infinite mass near some point u_0 in X. Assume that the matrix $(s, y, u) \to \gamma[c(s, y, \cdot)](u)$ is continuous on a neighborhood of $(0, x, u_0)$ and invertible at $(0, x, u_0)$. Assume moreover that it satisfies the following (local) ellipticity assumption:

$$\forall (s', x, u) \in]0, s] \times \mathbb{R}^d \times \mathcal{O},\ \gamma[c(s', x, u)] \geq \frac{1}{1 + |x|^\delta} \Theta(u) I_d,$$

Where $\delta, s > 0$ are constant, $\mathcal{O}$ is a neighborhood of u_0 and $\Theta > 0$ s.t.

$$\left(\int_0^s \int_{\mathcal{O}} \Theta(u)\, N(ds, du) \right)^{-1} \in \bigcap_{p \geq 2} L^p(\mathbb{P}). \quad (*)$$

Then, for all $t \geq s$ the solution X_t of (5) admits a density in $C_b^\infty(\mathbb{R}^d)$.

Lemma

Consider $\mathcal{O}$ and Θ as above and assume that there exists $\alpha \in (0,1)$ such that the limit

$$r_1 = \lim_{\lambda \to +\infty} \frac{1}{\lambda^\alpha} \int_{\mathcal{O}} (e^{-\lambda \Theta(u)} - 1)\, \nu(du)$$

exists and belongs to $(-\infty, 0)$ then hypothesis $(*)$ of the previous proposition is fulfilled.

Examples in the case where $\dim \Xi = +\infty$

- $\Xi = \mathbb{R}^+ \times C_0(\mathbb{R}^+;\mathbb{R}^q)$ where $q \in \mathbb{N}^*$ and $W = C_0(\mathbb{R}^+;\mathbb{R}^q)$ denotes the set of $\mathbb{R}^q$-valued continuous functions defined on $\mathbb{R}^+$ and vanishing in 0.
- $\nu = \tau \times m$ where m is the Wiener measure on $C_0(\mathbb{R}^+;\mathbb{R}^q)$ and τ is a Lévy measure on $\mathbb{R}^+$ associated to a subordinator such that $\tau(\mathbb{R}^+) = +\infty$. For simplicity, we assume that the support of τ is included in $[0, T']$ for some $T' > 0$.
- The Dirichlet structure $(\mathbf{d}, e, \gamma)$ is the product of the trivial structure on $L^2(R^+, \tau)$ with $(\mathbf{d}_M, e_M, \gamma_M)$, the Dirichlet structure on $L^2(C_0(\mathbb{R}^+;\mathbb{R}^q), m)$ associated with the Ornstein-Uhlenbeck operator.

As usual, we denote by $(B_t)_{t\geq 0}$ the coordinates maps on W:

$$\forall \omega \in W,\ B_t(\omega) = \omega_t,$$

so that $(B_t)_{t\geq 0}$ is a q-dimensional Brownian motion under the probability m.

A very simple example

Assume $d = q = 2$ and consider for all $t \geq 0$:

$$X_t = \begin{pmatrix} \int_0^t \int_0^{T'} \int_W B_y^1(\omega) N(ds, dy, d\omega) \\ \frac{1}{2} \int_0^t \int_0^{T'} \int_W (B_y^1(\omega))^2 N(ds, dy, d\omega) \end{pmatrix},$$

By the functional calculus, we have for all $t \geq 0$:

$$\Gamma[X_t] = \int_0^t \int_\Xi \gamma_M[B_y^1, \frac{1}{2}(B_y^1)^2](\omega) N(ds, dy, d\omega).$$

It is standard that

$$\gamma_M[B_y^1, \frac{1}{2}(B_y^1)^2] = \begin{pmatrix} y & yB_y^1 \\ yB_y^1 & y(B_y^1)^2 \end{pmatrix}.$$

Lemma

Assume that there exists $\alpha \in (0,1)$ such that the limit

$$r_1 = \lim_{\lambda \to +\infty} \frac{1}{\lambda^\alpha} \int_0^{T'} (e^{-\lambda y^2} - 1)\, \tau(dy)$$

exits and belongs to $(-\infty, 0)$ then X_t admits a density which belongs to $C_b^\infty(\mathbb{R}^2)$.

Remark: The hypothesis of the Lemma is fulfilled if for example $\tau(dy) = \frac{1}{y^{1+\epsilon}} dy$ with $\epsilon \in (0,1)$.

Remark: This is a very particular example of what we have called *non-linear subordination*.

- BICHTELER K., GRAVEREAUX J.-B., JACOD J. *Malliavin Calculus for Processes with Jumps* (1987).
- COQUIO A. "Formes de Dirichlet sur l'espace canonique de Poisson et application aux équations différentielles stochastiques" *Ann. Inst. Henri Poincaré* vol 19, n1, 1-36, (1993)
- FOURNIER N., GIET J.-S. "Existence of densities for jumping S.D.E.s" *Stoch. Proc. and Their Appl.* 116, 4(2005), 643-661.
- ISHIKAWA Y. and KUNITA H. "Malliavin calculus on the Wiener-Poisson space and its application to canonical SDE with jumps" *Stoch. Processes and their App.* 116, 1743-1769, (2006).
- LÉANDRE R. "Régularité de processus de sauts dégénérés (I), (II)" *Ann. Inst. Henri Poincaré* **21**, (1985) 125-146; **24** (1988), 209-236.

LÉANDRE R. "Regularity of degenerated convolution semi-groups without use of the Poisson space" preprint Inst. Mittag-Leffler (2007).

NOURDIN I. and SIMON T., "On the absolute continuity of Lévy processes with drift", *Ann. Probab.* 34 (2006), 1035-1051.

PICARD J."On the existence of smooth densities for jump processes" *Probab. Theory Relat. Fields* 105, 481-511, (1996)