Minimal Variance Hedging in incomplete markets: stochastic differentiation and the Clark-Ocone formula

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Outlines

1. Minimal variance hedging problem
2. Brownian motion
3. The compensated Poisson random measure.
4. Lévy martingales
5. Malliavin derivative for measures with independent values
6. MVH in a general martingale setting

1. Minimal variance hedging problem

Framework.

- Complete probability space $(\Omega, \mathcal{F}, P)$
- Fixed time horizon $[0, T]$, $T > 0$
- Right continuous filtration $\mathbb{F} := \{\mathcal{F}_t\,,\ t \in [0, T]\}$ representing the flow of information available in the market. For semplicity, $\mathcal{F}_0$ is trivial (up to P-null events) and $\mathcal{F} = \mathcal{F}_T$
- Market is frictionless, admitting short-selling and continuous trading
- $\eta = \eta_t$, $t \in [0, T]$, are the *discounted* prices of risky assets, modeled by a P-semimartingale. Assume $E\left[\eta_t^2\right] < \infty$. All financial claims and processes we consider are meant discounted.
- Ψ denotes the set of $\mathbb{F}$-predictable processes $\varphi = \varphi_t$, $t \in [0, T]$, representing the self-financing strategies via the holdings in the risky assets together with the initial capital x. The final value is:

$$V_T(x, \varphi) = x + \int_0^T \varphi_t d\eta_t \quad \in L_2(P).$$

MVH - Minimal variance hedging problem
Mean-variance hedging problem.

Let $\xi \in L_2(P)$ be a given (discounted) claim.

- Given x, find the best approximation $\hat{\xi}$ of ξ such that:
  - (i) there exists $\hat{\varphi} \in \Psi$ such that $\hat{\xi} = V_T(x, \hat{\varphi})$ (i.e. hedgeable)
  - (ii) $min_{\varphi \in \Psi} E\Big[|V_T(x,\varphi) - \xi|^2\Big] = E\Big[|V_T(x,\hat{\varphi}) - \xi|^2\Big]$
- Find explicit representations for the minimal variance hedging strategy $\hat{\varphi}$.

Ref. e.g. [S 1992, 2001, 2010]

From semimartingales to martingales.

The idea is to apply an adequate change of measure that preserves in some sense the orthogonality structure of the martingale part of η. The major suggestion is the *variance-optimal martingale measure* $\hat{P}$.

In this setting one can see that, in the modified MVH problem:

$$min_{x\in\mathbb{R};\varphi\in\Psi} E\Big[\big|V_T(x,\varphi)-\xi\big|^2\Big] = E\Big[\big|V_T(\hat{x},\hat{\varphi})-\xi\big|^2\Big],$$

the optimal initial capital is $\hat{x} = E_{\hat{P}}\big[\xi\big]$.

- If η is continuous, then $\hat{P}\sim P$. The density $\frac{d\hat{P}}{dP}$ can be exploited to give some more explicit representation of $\hat{\varphi}$. Ref. e.g. [S 1996], [DS 1996]
- If η is discontinuous, then no direct answer in the same context. The optimal strategy can be found as *locally risk-minimizing strategy*. The techniques requires another adequate change of measure P^* (*opportunity neutral*). The knowledge of $\frac{dP^*}{dP}$ can be used to give some form of representation of $\hat{\varphi}$. Ref. e.g. [S 2001], [CK 2007]
- N.B. The explicit form of such Radon-Nykodim derivatives is not obvious.

In a martingale setting.
Let η be already a P-martingale in L_2-setting.
This means that we are already in a *risk-neutral framework*.

The MVH problem. Find the best approximation $\hat{\xi}$ of ξ and the optimal strategy $\hat{\varphi}$ such that:

(i) there exists $\hat{\varphi} \in \Psi$ such that $\hat{\xi} = V_T(x, \hat{\varphi})$ (i.e. hedgeable)

(ii) $min_{\varphi \in \Psi} E\Big[|V_T(x,\varphi) - \xi|^2\Big] = E\Big[|V_T(x,\hat{\varphi}) - \xi|^2\Big]$

where $x = E\big[\xi\big]$.

Ref. e.g. [BL 1989], [FS 1986]

The solutions.

- $\hat{\xi}$ is found as the *projection* of ξ onto the subspace of $L_2(P)$ of stochastic integrals with respect to η.
  See Galtchouk-Kunita-Watanabe decomposition.
  The Itô isometry guarantees the *existence* (and *uniqueness*) of $\hat{\varphi}$:

$$\xi = \xi^0 + \hat{\xi} = \xi^0 + \int_0^T \hat{\varphi}_t d\eta_t. \tag{1}$$

  NB: $\xi^0 \in L_2(P)$ is in stochastic and it represents the non-hedgeable part of the claim in the MHV sense.

- Key problem: explicit representations of $\hat{\varphi}$.
  Use of *stochastic differentiation* and *CO formula*!

2. Brownian motion

Let $\eta_t = \sigma dW_t$ and let $\mathcal{F}_t = \mathcal{F}_t^W$ be generated by W.

Stochastic integral representation theorem.
Any $\xi \in L_2(P)$ can be represented as

$$\xi = E\big[\xi\big] + \int_0^T \varphi_t dW_t = E\big[\xi\big] + \int_0^T \frac{\varphi_t}{\sigma} d\eta_t$$

NB: The market is complete: all claims are hedgeable.

CO formula.
Any $\xi \in \mathbb{D}_{1,2}^W$ can be represented as

$$\xi = E\big[\xi\big] + \int_0^T E\big[D_t\xi|\mathcal{F}_t\big] dW_t.$$

- The solution to the MVH problem is the *perfect hedge*: $\hat{\xi} = \xi$ and

$$\hat{\varphi}_t = \frac{1}{\sigma} E\big[D_t\xi|\mathcal{F}_t\big].$$

NB: [OK 1991] Representation of the result under a general measure P.

3. The compensated Poisson random measure.

Let $\eta_t = \int_0^t \int_{\mathbb{R}_0} z\tilde{N}(dt, dz)$ with $m_2 := \int_{\mathbb{R}_0} z^2 \nu(dz) < \infty$ ($\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$).
Let $\mathcal{F}_t = \mathcal{F}_t^{\tilde{N}} = \sigma\{\tilde{N}\big((s,t], B\big) :\ 0 \le s \le t, B \in \mathcal{B}(\mathbb{R}_0)\}$.

Stochastic integral representation theorem.
Any $\xi \in L_2(P)$ can be represented as

$$\xi = E\big[\xi\big] + \int_0^T \int_{\mathbb{R}_0} \varphi(t,z)\tilde{N}(dt, dz).$$

NB: The market is incomplete: not all claims are hedgeable!
Example: Using Itô formula,

$$\begin{aligned} \xi = \eta_T^2 = \quad & E[\xi] + \int_0^T \int_{\mathbb{R}_0} z^2 \tilde{N}(dt, dz) + \int_0^T 2\eta_t d\eta_t \\ \neq \quad & E[\xi] + \int_0^T \varphi_t d\eta_t, \qquad \text{for some } \varphi \in \Psi. \end{aligned}$$

Comments.

- From Itô (1956) and Dellacherie (1974) we know that:
  Any $\xi \in L_2(P)$ admits integral representation in the form

  $$\xi = E\left[\xi\right] + \int_0^T \int_{\mathbb{R}_0} \theta_t d\eta_t$$

  when the integrator is either a Brownian motion or a centered Poisson process (easily extendable to Gaussian random field and compensated Poisson random measures).

  Question: For which stochastic measures $d\eta_t$ it is possible that *all* square integrable random variables admit a representation of the form above?

  Result. Among all the stochastic measures derived from stationary processes with independent increments, only the Gaussian and the centered Poisson ones have this property. (Generalized to homogeneous random fields with independent values).

  See [dN 2002b, 2007a].

Assume that there exists $\varepsilon > 0$ and $\delta > 0$ such that:

$$\int_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)} e^{\delta|z|}\nu(dz) < \infty \quad \text{and } m_k := \int_{\mathbb{R}_0} z^k \nu(dz),\ k > 1.$$

CO formula.

Any $\xi \in \mathbb{D}^{\tilde{N}}_{1,2}$ can be represented as

$$\xi = E\left[\xi\right] + \int_0^T \int_{\mathbb{R}_0} E\left[D_{t,z}\xi | \mathcal{F}_t\right] \tilde{N}(dt, dz)$$

- The solution to the MVH problem is then given by:

$$\hat{\varphi}_t = \frac{1}{m_2} \int_{\mathbb{R}_0} E\left[D_{t,z}\xi | \mathcal{F}_t\right] z\, \tilde{N}(dt, dz).$$

Example: For $\xi = \eta_T^2$ we have

$$\hat{\varphi}_t = \frac{1}{\int_{\mathbb{R}_0} z^2 \nu(dz)} \int_{\mathbb{R}_0} \left(2z^2 E\left[\eta_T | \mathcal{F}_t\right] + z^3\right)\nu(dz) = 2\eta_t + \frac{m_3}{m_2}.$$

4. Lévy martingales

Let

$$\eta_t = \sigma W_t + \int_0^t \int_{\mathbb{R}_0} z\tilde{N}(dt, dz)$$

with $\lambda := \sigma^2 + \int_{\mathbb{R}_0} z^2 \nu(dz) < \infty$.
Let $\mathcal{F}_t = \mathcal{F}_t^{W,\tilde{N}} = \mathcal{F}_t^W \vee \mathcal{F}_t^{\tilde{N}}$.

Stochastic integral representation theorem.
Any $\xi \in L_2(P)$ can be represented as

$$\xi = E[\xi] + \int_0^T \varphi_t^W dW_t + \int_0^T \int_{\mathbb{R}_0} \varphi^{\tilde{N}}(t, z) d\tilde{N}(dt, dz).$$

NB: The market is incomplete!

Wiener-Poisson product spaces

- One can consider

$$(\Omega, \mathcal{F}, P) = (\Omega^W, \mathcal{F}_T^W, P^W) \times (\Omega^{\tilde{N}}, \mathcal{F}_T^{\tilde{N}}, P^{\tilde{N}})$$

  with $P(d\omega_1, \omega_2) = P^W(\omega_1)P^{\tilde{N}}(\omega_2)$ and where

$$W_t(\omega_1, \omega_2) = W_t(\omega_1)$$

$$\tilde{N}(dt, dz; \omega_1, \omega_2) = \tilde{N}(dt, dz; \omega_2).$$

  Moreover, $\mathcal{F}_t = \mathcal{F}_t^{W,\tilde{N}}$.

In this case we can study a Malliavin calculus separately the Brownian case and the compensated Poisson random measure and merge the results.

Mixture of stochastic measures

- One can consider a stochastic measure μ on $[0, T] \times \mathbb{R}$ with independent values in $L_2(P)$ of the type:

$$\mu(dt, dz) := 1_{\{z=0\}}\mu^W(dt, dz) + 1_{\{z\in\mathbb{R}_0\}}\mu^{\tilde{N}}(dt, dz)$$

with $E\left[\mu(dt, dz)^2\right] = \delta_{\{0\}}(dz)dt + 1_{\{z\in\mathbb{R}_0\}}\nu(dz)dt$. This is the mixture of two *independent* Gaussian and compensated Poisson types of measures.

The filtration of information is $\mathcal{F}_t = \mathcal{F}_t^\mu = \mathcal{F}_t^{\mu^W, \mu^{\tilde{N}}}$

NB: Note that, with formal modifications, one can also use

$$\mu(dt, dz) := 1_{\{z=0\}}\mu^W(dt, dz) + z1_{\{z\in\mathbb{R}_0\}}\mu^{\tilde{N}}(dt, dz)$$

with $E\left[\mu(dt, dz)^2\right] = \delta_{\{0\}}(dz)dt + z^2 1_{\{z\in\mathbb{R}_0\}}\nu(dz)dt$.

NB: We can apply some chaos expansions in which the components of the two independent measures are mixed to derive. See [BdNLØP, 2003].

Generalized CO formula

Any $\xi \in \mathbb{D}_{1,2}$ (well defined within the two approaches) can be represented as:

$$\xi = E[\xi] + \int_0^T \int_{\mathbb{R}_0} E[D_t\xi|\mathcal{F}_t]\mu^W(dt,dz) + \int_0^T \int_{\mathbb{R}_0} E[D_{t,z}\xi|\mathcal{F}_t]\mu^{\tilde{N}}(dt,dz).$$

See [BdNLØP, 2003], [dNØP, 2009].

MVH problem.

The best approximation $\hat{\xi}$ to ξ in terms of minimal variance is:

$$\hat{\xi} = E[\xi] + \int_0^T \hat{\varphi}_t d\eta_t$$

with

$$\hat{\varphi}_t = \frac{1}{\lambda}\Big[\sigma E[D_t\xi|\mathcal{F}_t] + \int_{\mathbb{R}_0} zE[D_{t,z}\xi|\mathcal{F}_t]\nu(dz)\Big],$$

where $\lambda := \sigma^2 + \int_{\mathbb{R}_0} z^2\nu(dz) < \infty$.

See [BdNLØP, 2003], [dNØP, 2009].

Comments.

- In [LSUV, 2002] another extension of the CO formula is given via chaos expansions. These are based on a decomposition of the Levy noise into power-jump martingales, see [NS, 2000]. A version of the Generalized CO formula is also obtained in the case $\mu^{\tilde{N}}$ is a centered Poisson process.
- From the applications to finance point of view the domain of the Malliavin derivative operator is too small. For example:

$$\xi = 1_{[a,b]} \quad \notin \mathbb{D}_{1,2}$$

in the case of the Gaussian part and also in the case of the compensated Poisson random measure part (depending on ν).

Then and extension of these results to the whole $L_2(P)$ is necessary. In [DØP, 2004], this is achieved by means of white noise analysis.

Ref: Bernt Øksendal's presentation.

5. Malliavin derivative for measures with independent values

We discuss a calculus for

$$\eta_t = \sigma_t W_t + \int_{\mathbb{R}_0} z \tilde{N}(dt, dz)$$

directly designed for $\eta(dt)$. Here $\tilde{N}(dt, dz) = N(dt, dz) - \nu_t(dz)dt$. Also generalized to stochastic measures in L_2 with independent values.

See [dN 2007], [dNR 2007].

Approach via chaos expansions.

- The basic elements are the p-order multilinear forms:

$$I(1_{\Delta_1} \cdots 1_{\Delta_p}) = \eta(\Delta_1) \cdots \eta(\Delta_p),$$

  $p = 0, 1, ...,$ where $\Delta_1, ..., \Delta_p$ are disjoint intervals in $[0, T]$. These are simple iterated integrals of $\eta(dt)$. Let $H \subseteq L_2(\mathcal{F}, P)$ be the subspace generated by these p-order multilinear forms.

- A stochastic derivative is defined for the polynomials of any order of type

$$\xi = pol(\eta(\Delta_1), ..., \eta(\Delta_m))$$

where $\Delta_1, ..., \Delta_p$ are disjoint intervals in $[0, T]$ (any polynomial of values of the measure admits such - not unique - a representation):
Result. For any polynomial ξ, a stochastic derivative can be defined as the limit

$$\mathcal{D}\xi := \lim_{n\to\infty} \sum_{k=1}^{K_n} E\Big(\xi \frac{\eta(\Delta_{kn})}{E\left(|\eta(\Delta_{kn})|^2 | \mathcal{F}_{]\Delta_{kn}[}\right)} \Big| \mathcal{F}_{]\Delta_{kn}[}\Big) 1_{\Delta_{kn}}(t), \quad t \in [0, T],$$

which exists in $L_2([0, T] \times \Omega)$. Here the sum is taken over some partitions of $[0, T]$ and the limit is taken for the vanishing mesh. Such a limit can be determined by the formula

$$\mathcal{D}\xi = \sum_{j=1}^{m} \frac{\partial}{\partial \eta(\Delta_j)} pol(\eta(\Delta_1), ..., \eta(\Delta_m)) 1_{\Delta_j}(t)$$
$$+ \sum_{j=1}^{m} \Big[\sum_{k>1} \frac{m_{k+1}}{k!} \frac{\partial^k}{\partial \eta(\Delta_j)^k} pol(\eta(\Delta_1), ..., \eta(\Delta_m)) \Big] 1_{\Delta_j}(t), \ t \in [0, T].$$

- The operator $\mathcal{D}$ on the multilinear forms is a closable linear operator. Its minimal closed extension is:

$$D\xi = \mathcal{D}\xi \ : \quad H \ni \xi \quad \Longrightarrow \quad D\xi \in L_2([0,T] \times \Omega).$$

In the case η is Gaussian or compensated Poisson, then $H = L_2(P)$ and $D\xi$ coincides with the corresponding Malliavin derivative.

Approach via differential operator.

- For any

$$\xi = F(\eta(\Delta_1), \dots, \eta(\Delta_m)),$$

where $F \in C^1(\mathbb{R}^m)$ and $\Delta_1, \dots \Delta_m$ are disjoint intervals in $[0, T]$, we write

$$\partial_k^z F := \begin{cases} \frac{\partial}{\partial \eta(\Delta_k)} F(\dots, \eta(\Delta_k), \dots), & z = 0, \\ \frac{1}{z}\big[F(\dots, \eta(\Delta_k) + z, \dots) - F(\dots, \eta(\Delta_k), \dots)\big], & z \neq 0. \end{cases}$$

- We define

$$\mathbf{D}_t \xi := \sum_{k=1}^m \Big[\partial_k^0 F \sigma_t^2 + \int_{\mathbb{R}_0} \partial_k^z F z^2 \nu_t(dz)\Big] \cdot 1_{\Delta_k}(t).$$

We assumed that

$$|||\mathbf{D}\xi|||^2 := \sum_{k=1}^m \iint_{\Delta_k} \Big[\|\partial_k^0 F\|^2 \sigma_t^2 + \int_{\mathbb{R}_0} \|\partial_k^z F\|^2 z^2 \nu_t(dz)\Big] \nu_t(dz) dt < \infty.$$

Note that $\|\mathbf{D}\xi\|_{L_2([0,T]\times\Omega)} \le |||\mathbf{D}\xi|||$.

- Let $dom\mathbf{D} \subseteq L_2(P)$ be the linear closure of all the elements ξ satisfying the condition above.

  Result. The linear operator $\mathbf{D}$:

  $$L_2(\Omega) \supseteq dom\mathbf{D} \ni \xi \implies \mathbf{D}\xi \in L_2([0,T] \times \Omega),$$

  is closed with domain $dom\mathbf{D}$ dense in $L_2(\Omega)$.

  NB: This operator is studied in connection with the definitions given in the chaos expansions approach.

MVH in a general martingale setting

Let the discounted prices η_t, $t \in [0, T]$, be a general martingale in L_2 with $E[\eta(dt)] = 0$ and $E[\eta(dt)^2] = M(dt)$.

In this general case we can give a representation of the MVH strategy by means of a non-anticipating derivative.

Result.

The *non-anticipating stochastic derivative* $\mathfrak{D}\xi$ is well defined for all the elements $\xi \in L_2(\Omega)$ and it can be represented as the limit

$$\mathfrak{D}\xi = \lim_{n\to\infty} \sum_{k=1}^{\kappa_n} E\left[\xi \frac{\eta(\Delta_{nk})}{M(\Delta_{nk})} \Big| \mathcal{F}_{s_{nk}}\right] 1_{\Delta_{nk}},$$

with convergence in $L_2([0, T] \times \Omega)$. The limit is taken for the mesh vanishing of some partitions of $[0, T]$.

Any $\xi \in L_2(\Omega)$ admits representation in terms of its derivative, i.e.

$$\xi = \xi^0 + \int_0^T \mathfrak{D}_t \xi \, d\eta_t.$$

Moreover, we have $\xi^0 \in L_2(\Omega)$: $\quad \mathfrak{D}\xi^0 = 0$.

Ref. [dN 2002a, 2002b, 2007b], [dNR 2007].

Comments.

- The non-anticipating derivative corresponds to the CO formula kernel in all cases in which this one is defined.
- The non-anticipating derivative is the dual of the Itô integral.
- The MVH strategy corresponds to $\hat{\varphi}_t = \mathfrak{D}\xi$.
- The concept is extended to martingale random fields. Applications to MVH in large financial markets. See [dNE 2010].

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