Regularization for the homogeneous Boltzmann equation without cutoff

with V. Bally

The homogeneous Boltzmann equation

$f_t(x, v) =$ density of particles with position $x \in \mathbb{R}^2$ and velocity $v \in \mathbb{R}^2$ at time $t \geq 0$ in a gas.

If $f_0(x, v) = f_0(v)$, then for all $t \geq 0$, $f_t(x, v) = f_t(v)$: the gas is *spatially homogeneous*.

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The only dynamic we consider: binary elastic collisions.

$$v, v_* \longrightarrow v', v'_*$$

with conservation of momentum and kinetic energy:

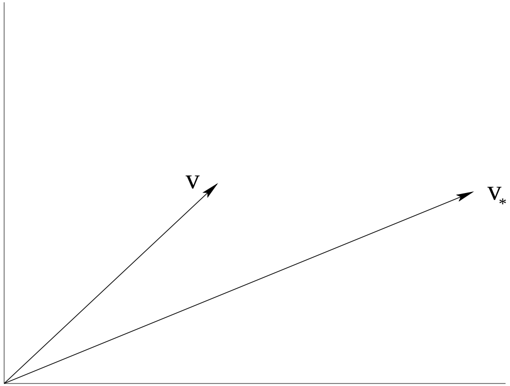
$$v' + v'_* = v + v_*$$

$$|v'|^2 + |v'_*|^2 = |v|^2 + |v_*|^2$$

This implies that for some $\theta \in (-\pi, \pi)$,

$$v' = v'(v, v_*, \theta) = \frac{v + v_*}{2} + R_\theta \frac{v - v_*}{2}$$

$$v'_* = v'_*(v, v_*, \theta) = \frac{v + v_*}{2} - R_\theta \frac{v - v_*}{2}$$



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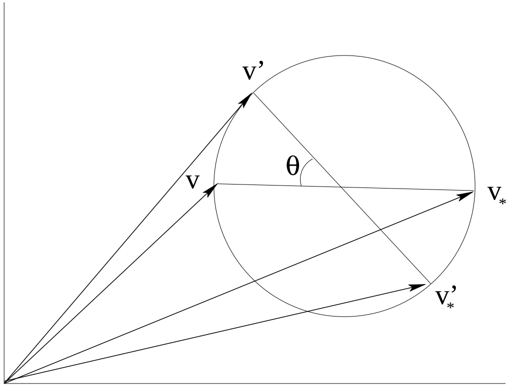

Collision rate:

$$v, v_* \xrightarrow{B(|v-v_*|,\theta)} v'(v, v_*, \theta), v'_*(v, v_*, \theta)$$

with

$$B(|v - v_*|, \theta) = \Phi(|v - v_*|)\beta(\theta).$$

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If the particles undergo a repulsion force in $1/r^s$, $s \in [2, \infty)$:

$$\Phi(|v - v_*|) = |v - v_*|^\gamma, \quad \gamma = \frac{s-5}{s-1} \in (-3, 1)$$

and

$$\beta(\theta) \overset{0}{\sim} \theta^{-1-\nu}, \quad \nu = \frac{2}{s-1} \in (0, 2).$$

We always have, if $s > 2$,

$$\int_{-\pi}^{\pi} \beta(\theta) d\theta = \infty, \quad \int_{-\pi}^{\pi} \theta^2 \beta(\theta) d\theta < \infty.$$

Each particle collides infinitely many other particles during each time interval.

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Many works on Boltzmann's eq. assume Grad's angular cutoff: $\int \beta(\theta) d\theta < \infty$.

Regularization is possible in the case without cutoff, and impossible in the case with cutoff. The first works on regularization for Boltzmann's equation are due to Desvillettes (around 1995).

We say that $(f_t)_{t\geq 0} \subset \mathcal{P}(\mathbb{R}^2)$ solves Boltzmann's equation if for all reasonnable $g : \mathbb{R}^2 \mapsto \mathbb{R}$,

$$\frac{d}{dt}\int_{\mathbb{R}^2} g(v) f_t(dv)$$
$$= \int_{\mathbb{R}^2} f_t(dv) \int_{\mathbb{R}^2} f_t(dv_*) \Phi(|v - v_*|) \int_{-\pi}^{\pi} \beta(\theta) d\theta [g(v') - g(v)]$$

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Villani (1999) has proved the existence of solutions in all the previously cited physical situations ($d = 3$).

With Mouhot, Guérin, (2005-2010), we obtained (sometimes global, sometimes local) uniqueness results in all the previously cited physical situations ($d = 3$).

Interpretation by a jumping SDE

Let us consider the **nonlinear** SDE with values in $\mathbb{R}^2$:

$$V_t = V_0 + \int_0^t \int_{\mathbb{R}^2} \int_{-\pi}^{\pi} \int_0^{\infty} A(\theta)(V_{s-} - v)\mathbf{1}_{\{u<|V_{s-}-v|^\gamma\}} N(ds, dv, d\theta, du),$$

where $A(\theta) = \begin{pmatrix} \cos\theta - 1 & -\sin\theta \\ \sin\theta & \cos\theta - 1 \end{pmatrix}$, and where N is a Poisson measure with intensity

$$ds\, \mathcal{L}(V_s)(dv)\, \beta(d\theta)\, du.$$

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Remark 1 : OK with $\int \theta^2 \beta(d\theta) < \infty$.

Remark 2 : $A(\theta)(V - v) = v'(V, v, \theta) - V$.

Regularization: results

Let us consider the case of a repulsive force in $1/r^s$, so that the interaction rate is $|v - v_*|^\gamma |\theta|^{-1-\nu}$ with $\gamma = (s-5)/(s-2)$ and $\nu = 2/(s-2)$. We assume that f_0 is a probability measure with an exponential moment and **is not a Dirac**.

- If $s > 24.14...$, then $f_t \in C_b(\mathbb{R}^2)\ \forall\ t > 0$.
- If $s > 13.75...$, then $f_t \in L^2(\mathbb{R}^2)\ \forall\ t > 0$.
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Some known results ($d = 3$):

- ADVW (2000): $f_t \in H^{\nu/2}_{loc}$ for any value of $s > 0$, if $\int f_0 |\log f_0| < \infty$.
- DW (2004): $f_t \in C^\infty$ if $(\epsilon^2 + |v - v_*|^2)^{\gamma/2}$ and if $\int f_0 |\log f_0| < \infty$.

The proof is based on Bally-Clément. BGJ doesn't apply (at all), because of the indicator function.

Let us recall that $f_t = \mathcal{L}(V_t)$ and

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Step 2: We study very finely $E[|V_t - V_t^{\epsilon,M}|]$ (and actually $E[|V_t - V_t^{\epsilon,M}|^\beta]$ for some well-chosen $\beta \in (0, 1]$).
Since we regularize $|V - v|^\gamma$ only on $|V - v| < \epsilon$, the estimate can be improved if we already know that f_t has some regularity.

Step 3: $V_t^{\epsilon,M}$ is a deterministic function of a (a.s. finite) family of r.v. $(T_n, V_n, U_n, \Theta_n)$. Using Bally-Clément's computations, we manage to do some IBP with respect to the r.v. Θ_n.

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Observe that the lowerbound of the Malliavin Covariance Matrix is non-trivial, since $(V_{s-} - v)$ may be 0. But we show that since f_0 is not a Dirac mass, f_t is not a Dirac mass, for all t, and thus we can always find some v's in the support of f_s such that $(V_{s-} - v)$ is nonzero. We quantify these facts rigorously.

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We find an estimate of the Fourier transform looking like: for all $q \geq 1$,

$$|E(e^{i\xi V_t^{\epsilon,M}})| \leq C_q |\xi|^{-q} M^{\nu q} \epsilon^{-q}.$$

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This is bad:

- when $\epsilon > 0$ is small (irreg. of $|V - v|^{\gamma}$)
- when M is large (irreg. of the indicator + many jumps).

And one needs to let $M \to \infty$ and $\epsilon \to 0$!!!

Step 4. We actually write

$$
\begin{aligned}
|\hat{f}_t(\xi)| = |E(e^{i\xi V_t})| \leq & |E(e^{i\xi V_t^{\epsilon,M}})| + |\xi| E[|V_t - V_t^{\epsilon,M}|] \\
\leq & C_q |\xi|^{-q} M^{\nu q} \epsilon^{-q} + |\xi| (\epsilon^{\nu+\gamma} + (1/M)^{1-\nu})
\end{aligned}
$$

At ξ fixed, we optimize this formula in q, ϵ and M. This gives

$$|\hat{f}_t(\xi)| \leq C|\xi|^{-q}.$$

for some $q > 0$ (when $s > 7$).

Step 5. This first regularity result allows us to obtain a better estimate for $E[|V_t - V_t^{\epsilon,M}|]$,
because $|\hat{f}_t(\xi)| \leq K|\xi|^{-\alpha}$ implies $f_t(B(v,\epsilon)) \leq C\epsilon^{\alpha}$, (valid when $\alpha \in (0,2)$.

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Thus we start again, and obtain a better value for q. And so on. In any case, at the end, we have something like

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And it holds that
- if $s > 24.14...$, then $|\hat{f}_t(\xi)| \leq C|\xi|^{-2-}$, thus $|\hat{f}_t(\xi)|$ is integrable, whence f_t has a continuous and bounded density;
- if $s > 13.75...$, then $|\hat{f}_t(\xi)| \leq C|\xi|^{-1-}$, thus $|\hat{f}_t(\xi)| \in L^2$, whence f_t has a density in L^2;
- if $s > 7$, then $q > 0$.

Another remark on regularization.

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Consider an interaction rate in $|v - v_*|^\gamma \beta(\theta)$, with $\gamma < 0$.

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All these results are based on the angular singularity: roughly, $v \mapsto v'(v, v_*, \theta)$, with rate $\beta(\theta)d\theta$. Since $\int \beta(\theta)d\theta = \infty$, each particle is immediately subjected to a collision, and the velocity distribution is immediately regular.

Assume that $d = 2$ and that f_0 is the uniform measure on the circle (with radius 1).

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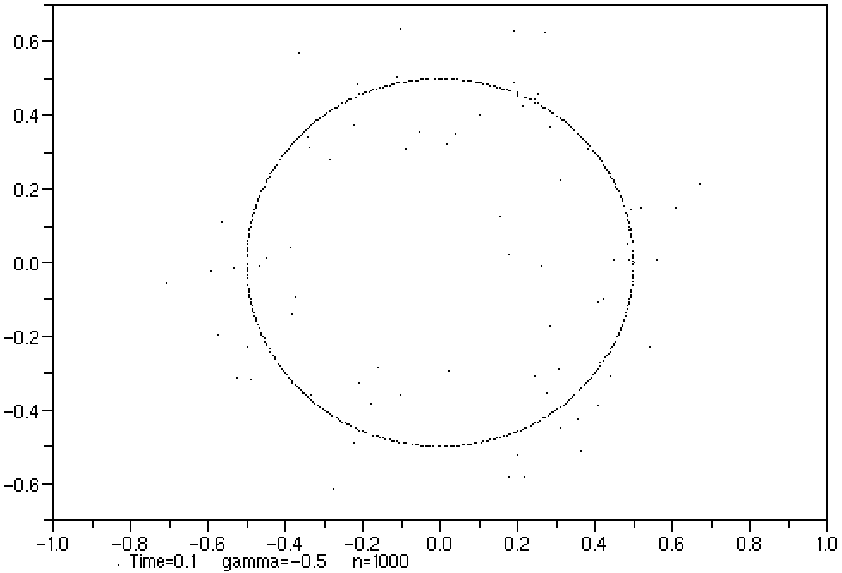
which is

- infinite (for f_0-a.a. v) if $\gamma < -1$, whence a possible regularization,
- finite (for all v) if $\gamma > -1$, whence no possible regularization.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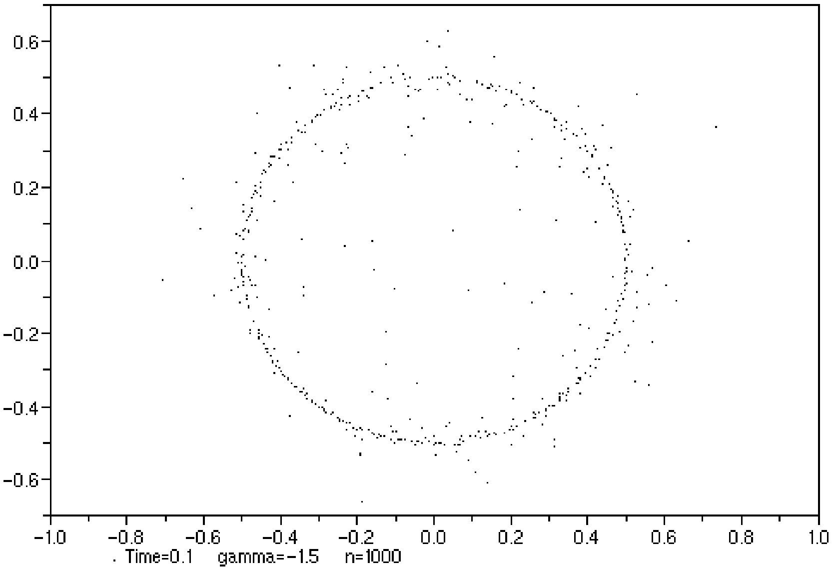
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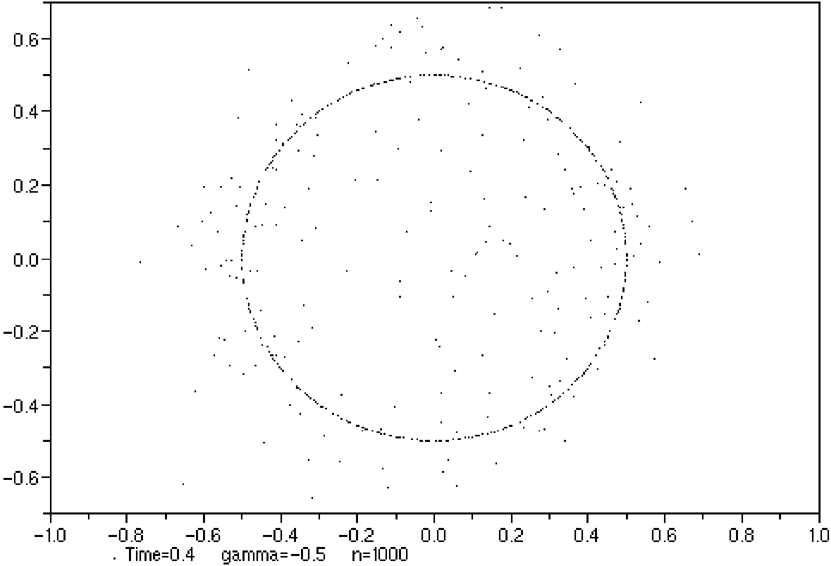
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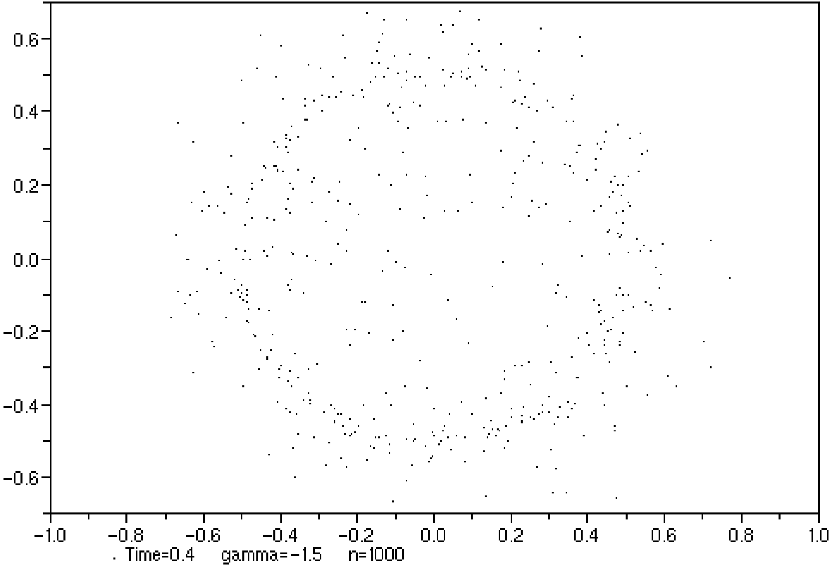
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We have a rigorous result for a 2d Botlzmann equation Linearized around its initial condition.