Malliavin Fractional Smoothness for Lévy Processes and Discrete Time Hedging

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1. Malliavin fractional smoothness
   - $\mathbb{D}_{1,2}$ via chaos expansion
   - Interpolation between $\mathbb{D}_{1,2}$ and L_2

2. Application in variance optimal hedging: discretization of the cumulative gains
   - Variance optimal hedging
   - Discretization of the cumulative gains
   - The martingale case
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3. Example: fractional smoothness of $1\!\!\mathrm{I}_{[K,\infty)}(S_1)$

4. References

1. Malliavin fractional smoothness

- $X = (X_t)_{t\in[0,1]}$ Lévy process with Lévy-Itô decomposition

$$X_t = \gamma t + \sigma W_t + \int_{(0,t]\times\{1<|x|\}} xN(ds,dx) + \int_{(0,t]\times\{0<|x|\leq 1\}} x\tilde{N}(ds,dx),$$

- (γ, σ^2, ν), ν Lévy measure.

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- **random measure** M

$$M(B) := \sigma \int_{\{t\in[0,1]:(t,0)\in B\}} dW_t + \lim_{n\to\infty} \int_{\{(t,x)\in B:1/n<|x|<n\}} x\,\tilde{N}(dt,dx).$$

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for all $B \in \mathcal{B}([0,1]\times\mathbb{R})$ such that

$$\int_B \mu(dx)dt := \sigma^2 \int_B \delta_0(dx)dt + \int_B x^2\nu(dx)dt < \infty$$

1.1. $\mathbb{D}_{1,2}$ via chaos expansion

- **chaos expansion** for $F \in L_2 := L_2(\Omega, \mathcal{F}^X, \mathbb{P})$

$$F = \sum_{m=0}^{\infty} I_m(f_m), \; a.s.$$

- $\mathbb{D}_{1,2} := \left\{ F \in L_2 : \|F\|_{1,2} := \sum_{m=0}^{\infty} (1+m) \| \; I_m(f_m) \|_{L_2}^2 < \infty \right\}$

1.2. Interpolation between $\mathbb{D}_{1,2}$ and L_2

- $K(u, F; L_2, \mathbb{D}_{1,2}) := \inf_{F=F_1+F_2}\{\|F_1\|_{L_2} + u\|F_2\|_{\mathbb{D}_{1,2}}\}, \quad u > 0$

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- Let $\theta \in (0,1)$, $q \in [1,\infty]$ and

$$\|F\|_{(L_2,\mathbb{D}_{1,2})_{\theta,q}} := \begin{cases} \left[\int_0^\infty [u^{-\theta} K(u, F; L_2, \mathbb{D}_{1,2})]^q \frac{du}{u}\right]^{\frac{1}{q}}, & q \in [1,\infty) \\ \sup_{u>0}\{u^{-\theta} K(u, F; L_2, \mathbb{D}_{1,2})\}, & q = \infty. \end{cases}$$

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Define

$$B^{\theta}_{2,q} := (L_2,\mathbb{D}_{1,2})_{\theta,q} := \{F \in L_2 : \|F\|_{(L_2,\mathbb{D}_{1,2})_{\theta,q}} < \infty\}$$

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- for $0<\theta_1<\theta_2<1$ and $q_1,q_2 \in [1,\infty]$

$$\mathbb{D}_{1,2} \subseteq B^{\theta_2}_{2,\min\{q_1,q_2\}} \subseteq B^{\theta_2}_{2,q_2} \subseteq B^{\theta_1}_{2,q_1} \subseteq L_2.$$

Lemma 1.1

For $\theta \in (0,1)$

$$B^{\theta}_{2,2} = \left\{ F \in L_2 : \|F\|^2_{B^{\theta}_{2,2}} = \sum_{m=0}^{\infty}(1+m^{\theta})\| \ I_m(f_m)\|^2_{L_2} < \infty. \right\}$$

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$$\begin{aligned} \mathbb{H} \ = \ \{ H = \sum_{m=0}^{\infty} I_m(f_m) \in L_2 : \\ f_m((t_1,x_1),\ldots,(t_m,x_m)) = \mathbb{1}^{\otimes m}_{(0,1]}(t_1,\ldots,t_m) g_m(x_1,\ldots,x_m) \} \end{aligned}$$

For example, $\mathbb{1}_{[0,\infty)}(X_1) \in \mathbb{H}$.

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Lemma 1.2 (Laukkarinen)

Let $\theta \in (0,1)$ *and* $H \in \mathbb{H}$. *Then it holds*

$$H \in B^{\theta}_{2,\infty} \quad \Longleftrightarrow \quad \begin{array}{l} \text{there exists a } c > 0 \\ \text{such that } \|H - \mathbb{E}[H|\mathcal{F}^X_t]\|^2_{L_2} \le c(1-t)^{\theta}. \end{array}$$

2. Application in variance optimal hedging: discretization of the cumulative gains

- assumption $\int_{\{|x|\geq 1\}} e^{2x}\nu(dx) < \infty$

$$\implies S_t := e^{X_t} \in L_2$$

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- martingale case: $\gamma_0 = -\frac{\sigma^2}{2} - \int_{\mathbb{R}} e^x - 1 - x\mathbb{1}_{\{|x|\leq 1\}}\nu(dx)$.

2.1. Variance optimal hedging

- predictable process (ϕ_t) (with integrability condition) and $c \in \mathbb{R}$ such that it holds

$$\mathbb{E}(H - c - G_1(\phi))^2 = \text{ minimal.}$$

- $G_1(\phi) := \int_{(0,1]} \phi_t dS_t$ denotes the cumulative gains.
- procedure to get (ϕ_t) and c :
  Föllmer-Schweizer decomposition

$$H = H_0 + \int_{(0,1]} \xi_t dS_t + L_1$$

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then

$$c = H_0$$

and

$$\phi_t = \xi_t + \frac{\delta}{\mu(\mathbb{R})}(H_{t-} - H_0 - G_{t-}(\phi)).$$

2.2. Discretization of the cumulative gains

Let $H \in \mathbb{H}$ and $\tau_N := \{0 = t_0 < t_1 < \ldots < t_N = 1\}$.

$$a(G_1(\phi); \tau_N) := \left\| \int_0^1 \phi_t dS_t - \sum_{k=1}^{N} \phi_{t_{k-1}} (S_{t_k} - S_{t_{k-1}}) \right\|_{L_2} \leq c N^{-r}$$

- best possible r ?
- r depends on the pattern of the (τ_N) and on the Malliavin fractional smoothness of $G_1(\phi)$.

2.3. The martingale case

Let $\gamma := \gamma_0$.

Theorem 2.1 (with S. Geiss, E. Laukkarinen)

Let $0 < \theta \le 1$ *and* $H \in \mathbb{H}$. *TFAE*

1. $G_1(\phi) \in B^{\theta}_{2,2}$.
2. *There exists a* $c > 0$ *such that for* $N = 1, 2, \ldots$

$$a(G_1(\phi); \tau_N^{\theta}) \le cN^{-\frac{1}{2}}$$

with

$$\tau_N^{\theta} := \left\{ t_k^{\theta} := 1 - \left(1 - \frac{k}{N}\right)^{\frac{1}{\theta}}, \quad k = 0, 1, \ldots, N \right\}.$$

Equidistant time nets: τ_N^1

Let $\gamma := \gamma_0$.

Theorem 2.2 (with S. Geiss, E. Laukkarinen)

Let $0 < \theta < 1$ *and* $H \in \mathbb{H}$. *TFAE*

1. *There exists a* $c > 0$ *such that for* $N = 1, 2, \ldots$
$$a(G_1(\phi); \tau_N^1)^2 \leq cN^{-\theta}$$
2. $G_1(\phi) \in B_{2,\infty}^{\theta}$.

Note that $B_{2,\infty}^{\theta} \subseteq B_{2,2}^{\eta} \quad \forall 0 < \eta < \theta$

2.4. The general case

$$\phi_t = \xi_t + \frac{\delta}{\mu(\mathbb{R})}(H_{t-} - H_0 - G_{t-}(\phi)).$$

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Föllmer-Schweizer decomposition

$$\begin{aligned} H &= H_0 + \int_{(0,T]} \xi_t dS_t + L_1 \\ &= H_0 + \delta \int_{(0,1]} \xi_t S_{t-} dt + \int_{(0,1]\times\mathbb{R}} \xi_t S_{t-} q(x) M(dt,dx) + L_1 \end{aligned}$$

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$\Longrightarrow$ BSDE $(Y_t, Z_{t,x})$

$$Y_t = H - \int_t^1 \frac{\delta}{\mu(\mathbb{R})} \int_{\mathbb{R}} Z_{s,y} q(y) d\mu(y) ds - \int_{(t,1]\times\mathbb{R}} Z_{s,x} M(ds,dx)$$

with $\xi_t S_{t-} = \frac{\delta}{\mu(\mathbb{R})} \int_{\mathbb{R}} Z_{t,y} q(y) d\mu(y)$

Theorem 2.3

Let $0 < \theta < 1$ *and* $H \in \mathbb{H}$. *TFAE*

1. $\int_0^1 \phi_t^{\gamma_0} dS_t^{\gamma_0} \in B_{2,\infty}^{\theta}$.
2. *Let* $\gamma \in \mathbb{R}$. *There exists a* $c > 0$ *such that for* $N = 1, 2, \ldots$

$$a(G_1^{\gamma}(\phi); \tau_N^1)^2 \leq cN^{-\theta}$$

3. Example: fractional smoothness of $\mathbb{1}_{[K,\infty)}(S_1)$

$\gamma = \gamma_0$, martingale case

(γ, σ^2, ν)	$\mathbb{1}_{[K,\infty)}(S_1)$
$\sigma > 0$ $\int_{\{\lvert x\rvert \geq 1\}} e^{2x}\nu(dx) < \infty$	$B^{\frac{1}{2}}_{2,\infty}$
Compound Poisson	$\mathbb{D}_{1,2}$
$(0, 0, \nu)$ Cauchy process: $d\nu(x) = \lvert x\rvert^{-2}dx$	$\mathbb{1}_{[K,\infty)}(S_1) \in B^{\theta}_{2,2}, \theta \in (0,1)$ and $\mathbb{1}_{[K,\infty)}(S_1) \notin \mathbb{D}_{1,2}$
$(0, 0, \nu)$ $d\nu(x) = \lvert x\rvert^{-2}\mathbb{1}_{[-1,1]}dx$	$\mathbb{1}_{[K,\infty)}(S_1) \in B^{\theta}_{2,2}, \theta \in (0,1)$ and $\mathbb{1}_{[K,\infty)}(S_1) \notin \mathbb{D}_{1,2}$

Assume $\int_{\{|x|\geq 1\}} e^{2x}\nu(dx) < \infty$.

Let $H = \mathbb{I}_{[K,\infty)}(S_1) = H_0 + G_1(\phi) + L_1$

stable like behavior for small jumps, $\sigma = 0$	Brodén Tankov	$G_1(\phi)$
with Blumenthal-Getoor index $\alpha \in (0, \frac{3}{2})$	$a(G_1(\phi); \tau_N^1)^2 \sim N^{-1}$	$\mathbb{D}_{1,2}$
$\alpha \in (\frac{3}{2}, 2)$	$a(G_1(\phi); \tau_N^1)^2 \sim N^{-\frac{3}{\alpha}-1}$	$B_{2,\infty}^{\frac{3}{\alpha}-1}$

$$\nu(dx) = \frac{f(x)}{|x|^{1+\alpha}}dx, \quad \lim_{x\downarrow 0} f(x) = f_+ > 0, \quad \lim_{x\uparrow 0} f(x) = f_- > 0$$

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