On Sobolev spaces of pure jump Lévy functionals

John J. A. Hosking, INRIA

INSTITUT NATIONAL
DE RECHERCHE
EN INFORMATIQUE
ET EN AUTOMATIQUE

INRIA

Talk for the Malliavin Calculus for Jump Processes workshop

Université Paris-Est

18 November 2010

This talk is principally based on work from my Ph.D. thesis which was conducted under the supervision of Professor Mark H. A. Davis at Imperial College London. I gratefully acknowledge the receipt of a Doctoral Training Award from the Engineering and Physical Sciences Research Council in that period.

Can the Picard operators, D and δ, be constructed as bounded linear operators on a family of Sobolev spaces of pure jump Lévy functionals (PJLFs) defined in an analogous form to Watanabe's Sobolev spaces from the Wiener functional case?

Watanabe's Sobolev Spaces [Wat83] [Wat87]

Abstract Wiener space $(W, H_W, \mathbb{P}_W)$.
Separable Hilbert space E.
Polynomial Wiener functionals $\mathcal{P}_W(E)$.
Ornstein-Uhlenbeck (OU) operator $\mathcal{L}_W$.

For any $p \in (1, \infty)$ and any $r \in \mathbb{R}$ define $\mathbb{D}_W^{p,r}(E)$ as the completion of $\mathcal{P}_W(E)$ with respect to the norm

$$\| \cdot \|_{\mathbb{D}_W^{p,r}(E)} := \| (\mathrm{Id} - \mathcal{L}_W)^{\frac{r}{2}} \cdot \|_{L^p(W;E)}.$$

For any $p \in (1,\infty)$ and any $r \in \mathbb{R}$ then it is known that D_W, D_W^*, $\mathcal{L}_W$ and C_W, can be extended from definitions on $\mathcal{P}_W(E)$ to the bounded linear operators

$$
\begin{aligned}
D_W &: \mathbb{D}_W^{p,r}(E) \to \mathbb{D}_W^{p,r-1}(H_W \otimes E) \\
D_W^* &: \mathbb{D}_W^{p,r}(H_W \otimes E) \to \mathbb{D}_W^{p,r-1}(E) \\
\mathcal{L}_W &: \mathbb{D}_W^{p,r}(E) \to \mathbb{D}_W^{p,r-2}(E) \\
C_W &: \mathbb{D}_W^{p,r}(E) \to \mathbb{D}_W^{p,r-1}(E).
\end{aligned}
$$

Theorem 1 (Krée-Meyer Inequalities (Mey84))**.** *For each* $p \in (1, \infty)$ *and* $k \in \mathbb{N}$ *there exists constants* $c, c' > 0$ *such that*

$$c\|D^k F\|_{L^p(W;E\otimes H_W^{\otimes k})} \leq \|C^k F\|_{L^p(W;E)} \leq c'(\|F\|_{L^p(W;E)} + \|D^k F\|_{L^p(W;E\otimes H_W^{\otimes k})})$$

for all $F \in \mathcal{P}_W(E)$*.*

Krée-Meyer inequalities provide a link between:

- Chaotic multiplier operators. Convenient for duality (e.g. ‘reflexivity’).
- Malliavin derivative and adjoint operators. Convenient for operations on functions of functionals (e.g. chain rule).

Motivation : Uses of the Watanabe's Sobolev spaces [Wat83] [Wat87]

- A theory of distributions for the Wiener space:
  - Malliavin calculus for generalized Wiener functionals
  - Composition of Schwartz distributions and non-degenerate smooth Wiener functionals
- Existence of probability densities
- Asymptotic expansions

2. Approaches to a Malliavin calculus for PJLFs: Set-up

We consider the following special case of the Picard [Pic96] set-up:

- Let ν be some given Lévy measure on $(\mathbb{R}_0^d, \mathcal{B}(\mathbb{R}_0^d))$, and define $\pi :=$ Leb $\otimes \nu$ on $([0,T] \times \mathbb{R}_0^d, \mathcal{B}([0,T] \times \mathbb{R}_0^d))$.

- Let $\mathcal{B}_0([0,T] \times \mathbb{R}_0^d) := \{A \times B \in \mathcal{B}([0,T] \times \mathbb{R}_0^d) | \pi(A \times B) < \infty\}$.

- There exists a $(\Omega^0, \mathbb{F}^0, \mathbb{P}^0)$ supporting a random Poisson measure μ^0 for $([0,T] \times \mathbb{R}_0^d, \mathcal{B}([0,T] \times \mathbb{R}_0^d))$ with intensity measure π, such that $\mu^0(\{t\} \times \mathbb{R}_0^d)(\omega) \in \{0,1\}$ for all $\mathbb{P}^0$-almost all $\omega \in \Omega^0$ uniformly for all $t \in [0,T]$.

2. Approaches to a Malliavin calculus for PJLFs: Set-up

- Define Ω as the set of all $\mathbb{N} \cup \{0\}$-valued measures ω on $([0,T] \times \mathbb{R}_0^d, \mathcal{B}([0,T] \times \mathbb{R}_0^d))$ such that:

  (i) $\omega(\{t\} \times \mathbb{R}_0^d) \in \{0,1\}$ for all $t \in [0,T]$; and

  (ii) $\omega(A \times B) < \infty$ for all $A \times B \in \mathcal{B}_0([0,T] \times \mathbb{R}_0^d)$.

- Define $\mathcal{F}'$ as the σ-algebra of subsets of Ω generated by the collection of mappings $\mu(A \times B) : \Omega \to \mathbb{N} \cup \{0\}$ where $\mu(\omega) := \omega$ for all $\omega \in \Omega$.

- Define $\mathbb{P}'$ as the image measure of $\mathbb{P}^0$ under the mapping $\mu^0 : \Omega^0 \to \Omega$. Hence under $\mathbb{P}'$ then μ is a Poisson random measure with intensity measure π.

- Define $\mathcal{F}$ as the completion of $\mathcal{F}'$ with the collection of $\mathbb{P}'$-negligible subsets of Ω.

- Let $\mathbb{P}$ be the completion of $\mathbb{P}'$ on $(\Omega, \mathcal{F})$.

It is random variables of this $(\Omega, \mathcal{F}, \mathbb{P})$ probability space that we refer to as pure jump Lévy functionals.

For each $m \in \mathbb{N}$ define a class $\mathcal{E}_m$ of elementary functions, mapping from $([0,T] \times \mathbb{R}_0^d)^m$ to $\mathbb{R}$, by such functions of the form

$$f(t_1, \ldots, t_m, z_1, \ldots, z_m) = \sum_{i_1,\ldots,i_m=1}^{n} \alpha_{i_1,\ldots,i_m} \prod_{j=1}^{m} \mathbf{1}_{A_{i_j} \times B_{i_j}}(t_j, z_j) \quad (1)$$

where $n \in \mathbb{N}$, $\{A_j \times B_j\}_{j=1}^n$ are disjoint subsets from $\mathcal{B}_0([0,T] \times \mathbb{R}_0^d)$, and the coefficients $\alpha_{i_1,\ldots,i_m}$ are zero whenever two or more of their indices match.

For any such $f \in \mathcal{E}_m$ of the form in equation (1) then $I_m[f]$, the m^{th} Poisson MSI of f, is defined as

$$I_m[f] := \sum_{i_1,\ldots,i_m=1}^{n} \alpha_{i_1,\ldots,i_m} \prod_{j=1}^{m} (\mu - \pi)(A_{i_j} \times B_{i_j}).$$

Each $I_m : \mathcal{E}_m \to L^2(\Omega; \mathbb{R})$ extends to a bounded linear operator $I_m : L^2_{\mathsf{sym}}(([0,T] \times \mathbb{R}^d_0)^m; \mathbb{R}) \to L^2(\Omega; \mathbb{R})$.

Theorem 2 (Itô (Itô56))**.**

$$L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}) = \bigoplus_{m=0}^{\infty} \mathcal{I}_m$$

where $\mathcal{I}_m := \{I_m[f] \in L^2(\Omega; \mathbb{R}) | f \in L^2_{\mathsf{sym}}(([0,T] \times \mathbb{R}^d_0)^m; \mathbb{R})\}$

Denote by P_m the projection $P_m : L^2(\Omega; \mathbb{R}) \to \mathcal{I}_m$.

Charlier polynomials

For parameter $\lambda > 0$. Functions

$$\{C_m(\cdot;\lambda) : \{n-\lambda\}_{n\in\mathbb{N}\cup\{0\}} \to \mathbb{R}\}_{m\in\mathbb{N}\cup\{0\}}$$

with a generating function

$$C(z,x,\lambda) := \sum_{m=0}^{\infty} \frac{z^m}{m!} C_m(x;\lambda) = (1+z)^{x+\lambda} e^{-z\lambda}.$$

Proposition 3 (Surgailis (Sur84))**.** *For any* $n \in \mathbb{N}$, $\{m_i\}_{i=1}^n \subset \mathbb{N} \cup \{0\}$ *and disjoint* $\{A_i \times B_i\}_{i=1}^n \subset \mathcal{B}_0([0,T] \times \mathbb{R}_0^d)$ *then*

$$I_{(\sum_{i=1}^n m_i)}[\odot_{i=1}^n (\mathbf{1}_{A_i \times B_i}^{\odot m_i})] = \prod_{i=1}^n C_{m_i}(I_1[\mathbf{1}_{A_i \times B_i}]; \pi(A_i \times B_i)).$$

Note, for any $m_1, m_2 \in \mathbb{N}$, $f_1 : ([0,T] \times \mathbb{R}_0^d)^{m_1} \to \mathbb{R}$ and $f_2 : ([0,T] \times \mathbb{R}_0^d)^{m_2} \to \mathbb{R}$ then define $f_1 \odot f_2 : ([0,T] \times \mathbb{R}_0^d)^{m_1+m_2} \to \mathbb{R}$ by $f_1 \odot f_2 := \mathrm{sym}(f_1 f_2)$.

Separable Hilbert space E, with complete orthonormal basis $\{\eta_i\}_{i\in\mathbb{N}}$

Define the set of E-valued polynomial PJLFs $\mathcal{P}(E)$ as the quotient space

$$\mathcal{P}(E) := \Big\{ \sum_{i=1}^{N} p_i((\mu-\pi)(A_1 \times B_1), \ldots, (\mu-\pi)(A_n \times B_n))\eta_i \\ \Big| n, N \in \mathbb{N}, \{A_i \times B_i\}_{i=1}^{n} \subset \mathcal{B}_0([0,T] \times \mathbb{R}_0^d) \text{ disjoint }, \\ \text{polynomials } p_i : \mathbb{R}^n \to \mathbb{R} \Big\} \setminus \sim_{\mathbb{P}} .$$

The Picard mass transformations $\varepsilon^-_{\cdot,\cdot}$, $\varepsilon^+_{\cdot,\cdot}$ are defined on any $\omega \in \Omega$ by

$$\begin{aligned} \varepsilon^-_{t,z}\omega(A \times B) &:= \omega((A \times B) \cap \{(t,z)\}^c) \\ \varepsilon^+_{t,z}\omega(A \times B) &:= \varepsilon^-_{t,z}\omega(A \times B) + \mathbf{1}_{A\times B}(t,z) \end{aligned}$$

for all $A \times B \in \mathcal{B}([0,T] \times \mathbb{R}^d_0)$ and for all $(t,z) \in [0,T] \times \mathbb{R}^d_0$.

The Picard difference operator $D_{\cdot,\cdot}$ is then defined, here for $\mathcal{P}(E)$, by

$$D_{t,z}F = F \circ \varepsilon^+_{t,z} - F$$

for π-almost all $(t,z) \in [0,T] \times \mathbb{R}^d_0$.

The Picard δ-operator is defined for $\mathbb{P} \otimes \pi$-integrable processes $Y : \Omega \times [0,T] \times \mathbb{R}_0^d \to \mathbb{R}$ by

$$\begin{aligned}\delta(Y) &:= \int_{[0,T]\times\mathbb{R}_0^d} Y(t,z) \circ \varepsilon_{t,z}^- \,(\mu - \pi)(\mathrm{d}\,t, \mathrm{d}\,z) \\ &= \int_{[0,T]\times\mathbb{R}_0^d} Y(t,z) \circ \varepsilon_{t,z}^- \,\mu(\mathrm{d}\,t, \mathrm{d}\,z) - \int_{[0,T]\times\mathbb{R}_0^d} Y(t,z)\,\pi(\mathrm{d}\,t, \mathrm{d}\,z).\end{aligned}$$

Let $H := L^2([0,T] \times \mathbb{R}_0^d, \mathcal{B}([0,T] \times \mathbb{R}_0^d), \pi; \mathbb{R})$ and $H_0 := \mathcal{E}_1 \subset H$.

Define $D : \mathcal{P}(E) \to \mathcal{P}(E \otimes H_0)$ and $D^* : \mathcal{P}(E \otimes H_0) \to \mathcal{P}(E)$ by

$$
\begin{aligned}
DF &:= \sum_{i=1}^{\infty} (D_{\cdot,\cdot} \langle F, \eta_i \rangle_E) \eta_i \\
D^* Y &:= \sum_{i=1}^{\infty} \delta(\langle Y(\cdot,\cdot), \eta_i \rangle_E) \eta_i
\end{aligned}
$$

for all $F \in \mathcal{P}(E)$ and $Y \in \mathcal{P}(E \otimes H_0)$.

It is known from Picard [Pic96] that these operators D and D^* must be such that

$$\mathbb{E}[\langle D^*(Y), F\rangle_E] = \mathbb{E}\Big[\langle Y, DF\rangle_{E\otimes H}\Big].$$

for all $F \in \mathcal{P}(E)$ and $Y \in \mathcal{P}(E \otimes H_0)$.

It is noted in [Pic96] that $D_{\cdot,\cdot}$ and δ are closable in their respective L^2-spaces—giving mutually dual closed linear operators

$$\begin{aligned} D_{\cdot,\cdot} &: \mathrm{Dom}_{L^2(\Omega;\mathbb{R})}(D_{\cdot,\cdot}) \to L^2(\Omega \times [0,T] \times \mathbb{R}_0^d; \mathbb{R}) \\ \delta &: \mathrm{Dom}_{L^2(\Omega\times[0,T]\times\mathbb{R}_0^d;\mathbb{R})}(\delta) \to L^2(\Omega; \mathbb{R}). \end{aligned}$$

Løkka [Løk04] has shown that this L^2-closure of the Picard difference operator, $D_{\cdot,\cdot} : \mathrm{Dom}_{L^2(\Omega;\mathbb{R})}(D_{\cdot,\cdot}) \to L^2(\Omega \times [0,T] \times \mathbb{R}_0^d)$, is equals the operator $\tilde{D} : \mathrm{Dom}_{L^2(\Omega;\mathbb{R})}(\tilde{D}) \to L^2(\Omega \times [0,T] \times \mathbb{R}_0^d)$ defined from the L^2-Poisson MSI chaos decomposition by

$$\tilde{D}_{t,z}F = \sum_{m=1}^{\infty} mI_{m-1}[f_m(\cdot,t,z)]$$

for any $F = \sum_{m=0}^{\infty} I_m[f_m] \in L^2(\Omega;\mathbb{R})$.

Define a Cauchy-type operator $C : \mathcal{P}(E) \to \mathcal{P}(E)$ by

$$CF := \sum_{i=1}^{\infty} \sum_{m=1}^{\infty} m^{1/2} P_m(\langle F, \eta_i \rangle_E)\eta_i$$

for any $F \in \mathcal{P}(E)$.

Lemma 4. *For any $F \in \mathcal{P}(E)$ then*

$$\|DF\|_{L^2(\Omega;E\otimes H)} = \|CF\|_{L^2(\Omega;E)}.$$

For each $k \in \mathbb{N}$, define $C^k : \mathcal{P}(E) \to \mathcal{P}(E)$ and $D^k : \mathcal{P}(E) \to \mathcal{P}(E \otimes H_0^{\odot k})$ inductively by $C^k := C \circ C^{k-1}$ and $D^k := D \circ D^{k-1}$.

Proposition 5. *For any $k \in \mathbb{N}$ there exists a constant $c_k > 0$ such that*

$$\begin{aligned}\|D^k F\|_{L^2(\Omega;E\otimes H^{\odot k})} &\leq \|C^k F\|_{L^2(\Omega;E)} \\ &\leq c_k \|F\|_{L^2(\Omega;E)} + c_k \|D^k F\|_{L^2(\Omega;E\otimes H^{\odot k})}\end{aligned}$$

for all $F \in \mathcal{P}(E)$.

Definition 6. *The family of norms* $\{\|\cdot\|_{\mathbb{D}^{2,r}_{\mathcal{L}}(E)} | r \in \mathbb{R}\}$ *on* $\mathcal{P}(E)$ *is defined for each* $r \in \mathbb{R}$ *by* $\|\cdot\|_{\mathbb{D}^{2,r}_{\mathcal{L}}(E)} := \|(\mathrm{Id} - \mathcal{L})^{\frac{r}{2}} \cdot \|_{L^2(\Omega;E)}$*, where* $(\mathrm{Id} - \mathcal{L})^{\frac{r}{2}} : \mathcal{P}(E) \to \mathcal{P}(E)$ *is the operator*

$$(\mathrm{Id} - \mathcal{L})^{\frac{r}{2}} := \sum_{i=1}^{\infty} \sum_{m=0}^{\infty} (1+m)^{\frac{r}{2}} P_m(\langle \cdot, \eta_i \rangle_E) \eta_i.$$

The family of Sobolev spaces $\{\mathbb{D}^{2,r}_{\mathcal{L}}(E) | r \in \mathbb{R}\}$ *are defined such that, for each* $r \in \mathbb{R}$*, then* $\mathbb{D}^{2,r}_{\mathcal{L}}(E)$ *is the Banach space completion of* $\mathcal{P}(E)$ *with respect to the norm* $\|\cdot\|_{\mathbb{D}^{2,r}_{\mathcal{L}}(E)}$.

Lemma 7. *Let* $r, r' \in \mathbb{R}$*.*

(i) If $r' \leq r$ *then* $\|F\|_{\mathbb{D}_{\mathcal{L}}^{2,r'}(E)} \leq \|F\|_{\mathbb{D}_{\mathcal{L}}^{2,r}(E)}$ *for all* $F \in \mathcal{P}(E)$*, hence* $\mathbb{D}_{\mathcal{L}}^{2,r}(E) \subseteq \mathbb{D}_{\mathcal{L}}^{2,r'}(E)$*.*

(ii) The norms $\|\cdot\|_{\mathbb{D}_{\mathcal{L}}^{2,r}(E)}$ *and* $\|\cdot\|_{\mathbb{D}_{\mathcal{L}}^{2,r'}(E)}$ *are compatible.*

(iii) $\mathbb{D}_{\mathcal{L}}^{2,r}(E) = (\mathbb{D}_{\mathcal{L}}^{2,-r}(E))^*$

Proposition 8. *For each $r \in \mathbb{R}$ there exist unique extensions of the operators*

$$D : \mathcal{P}(E) \to \mathcal{P}(E \otimes H_0)$$
$$D^* : \mathcal{P}(E \otimes H_0) \to \mathcal{P}(E)$$
$$\mathcal{L} : \mathcal{P}(E) \to \mathcal{P}(E)$$
$$C : \mathcal{P}(E) \to \mathcal{P}(E)$$

to the bounded linear operators

$$D : \mathbb{D}_{\mathcal{L}}^{2,r}(E) \to \mathbb{D}_{\mathcal{L}}^{2,r-1}(E \otimes H)$$
$$D^* : \mathbb{D}_{\mathcal{L}}^{2,r}(E \otimes H) \to \mathbb{D}_{\mathcal{L}}^{2,r-1}(E)$$
$$\mathcal{L} : \mathbb{D}_{\mathcal{L}}^{2,r}(E) \to \mathbb{D}_{\mathcal{L}}^{2,r-2}(E)$$
$$C : \mathbb{D}_{\mathcal{L}}^{2,r}(E) \to \mathbb{D}_{\mathcal{L}}^{2,r-1}(E).$$

Proofs of the Krée-Meyer inequalities:

- P.-A. Meyer [Mey84]
- R. Gundy [Gun86]
- G. Pisier [Pis88]

All rely in some way on the hypercontractivity of the OU semigroup and/or the equivalence between L^p and H^p.

Define semigroup $\{T_t : L^2(\Omega; E) \to L^2(\Omega; E)\}$ by

$$T_t F := \sum_{i=1}^{\infty} \sum_{m=0}^{\infty} \exp\{-mt\} P_m(\langle F, \eta_i \rangle_E) \eta_i$$

We call this the quasi Ornstein-Uhlenbeck (QOU) semigroup (or rather its $L^2(\Omega; E)$-case). It is

- E-conservative.
- Symmetric in $\langle \cdot, \cdot \rangle_{L^2(\Omega; E)}$.
- Strongly continuous semigroup of $L^2(\Omega; E)$-contraction operators

- From Surgailis [Sur83] we know that this QOU semigroup will be positivity preserving.

- Thus for each $p \in (1, \infty)$, one can extend/restrict the QOU semigroup to form an E-conservative semigroup of bounded linear operators $\{T_t : L^p(\Omega; E) \to L^p(\Omega; E)\}_{t \in [0,\infty)}$.

- For each $p \in (1, \infty)$ the corresponding QOU semigroup is strongly continuous and defines a generator $\mathcal{L} : \mathrm{Dom}_{L^p(\Omega;E)}(\mathcal{L}) \to L^p(\Omega; E)$ (closed and densely defined).

- On $\mathcal{P}(E)$ each $\mathcal{L} : \mathrm{Dom}_{L^p(\Omega;E)}(\mathcal{L}) \to L^p(\Omega; E)$ agrees with $\mathcal{L} : \mathcal{P}(E) \to \mathcal{P}(E)$.

Proposition 9 (c.f. Surgailis (Sur84))**.** *The QOU semigroup* $\{T_t\}_{t\in[0,\infty)}$ *is not hypercontractive.*

Proof. ($E=\mathbb{R}$ case) For any $A\times B\in\mathcal{B}_0([0,T]\times\mathbb{R}_0^d)$ let $F:=I_1[\mathbf{1}_{A\times B}]=(\mu-\pi)(A\times B)\in L^2(\Omega;\mathbb{R})$, then

$$\begin{aligned}\|F\|_{L^2(\Omega;\mathbb{R})} &= \pi(A\times B)^{1/2}\\ \|T_tF\|_{L^4(\Omega;\mathbb{R})} &= e^{-t}(3\pi(A\times B)^2+\pi(A\times B))^{1/4}.\end{aligned}$$

Hypercontractivity would require that there exists constants $c>0$ and $t\in[0,\infty)$ such that $\|T_tF\|_{L^4(\Omega;R)}\le c\|F\|_{L^2(\Omega;\mathbb{R})}$, hence that

$$3+\pi(A\times B)^{-1}\le ce^t$$

for any $A\times B\in\mathcal{B}_0([0,T]\times\mathbb{R}_0^d)$. For a contradiction: just let $A\times B$ have small enough π-measure. □

Same idea gives a counterexample to the general Krée-Meyer type inequality in this PJLF setting.

Proposition 10. *For any constant* $c > 0$ *there exists a* $F \in \mathcal{P}(\mathbb{R})$ *such that*

$$c\|DF\|_{L^4(\Omega;H)} < \|CF\|_{L^4(\Omega;\mathbb{R})}.$$

By duality: for any constant $c > 0$ there exists a $F \in \mathcal{P}(\mathbb{R})$ such that

$$c\|CF\|_{L^{4/3}(\Omega;H)} < \|DF\|_{L^{4/3}(\Omega;H)}.$$

Furthermore, the following properties are **false**:

- for all $p \in (1, \infty)$ there exists constants $c_1, c_2 > 0$ such that
  $$c_1 \|CF\|_{L^p(\Omega;\mathbb{R})} \leq \|F\|_{L^p(\Omega;\mathbb{R})} + \|DF\|_{L^p(\Omega;H)} \leq c_2 \|CF\|_{L^p(\Omega;\mathbb{R})}$$
  for all $F \in \mathcal{P}(\mathbb{R})$.

- On any fixed finite order Poisson MSI chaos any two norms in $\{\| \cdot \|_{L^p(\Omega;\mathbb{R})} | p \in (1, \infty)\}$ are equivalent.

- For each $p \in (1, \infty)$, then L^p and an analogous H^p for the setting defined with π, are equivalent.

Also see Lenglart, Lépingle and Pratelli [LLP80].

Suppose

- $(\Omega', \mathcal{F}', \mathbb{P}')$ and $X(\epsilon) \sim D(\epsilon)$ with finite moments of all orders and $\mathbb{E}_{\mathbb{P}'}[X(\epsilon)] = 0$.

- $\{P_i^{(\epsilon)}\}_{i \in \mathbb{N} \cup \{-1,0\}}$ monic orthogonal polynomials for $D(\epsilon)$, $P_{-1}^{(\epsilon)}(x) := 1$, $P_0^{(\epsilon)}(x) := 1$,
$$P_{n+1}^{(\epsilon)}(x) = (x - a_n(\epsilon))P_n^{(\epsilon)}(x) - b_n(\epsilon)P_{n-1}^{(\epsilon)}(x)$$
for all $n \in \mathbb{N} \cup \{0\}$.

- $\mathbb{E}_{\mathbb{P}'}[P_n^{(\epsilon)}(X(\epsilon))^2] = c_n \sigma(\epsilon)^n$ for all $n \in \mathbb{N}$, where $\sigma(\epsilon) \downarrow 0$ as $\epsilon \downarrow 0$.

Note that: $a_0(\epsilon) = 0$, $a_1(\epsilon) = c_1^{-1}\sigma(\epsilon)^{-1}\mathbb{E}_{\mathbb{P}'}[X(\epsilon)^3]$, $b_1(\epsilon) = c_1\sigma(\epsilon)$, $P_1^{(\epsilon)}(x) = x$, and $P_2^{(\epsilon)}(x) = x(x - a_1(\epsilon)) - c_1\sigma(\epsilon)$. Hence

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}'}[X(\epsilon)^4] &= \mathbb{E}_{\mathbb{P}'}[(P_2^{(\epsilon)}(X(\epsilon)) + a_1(\epsilon)P_1^{(\epsilon)}(X(\epsilon)) + c_1\sigma(\epsilon))^2] \\
&= c_2\sigma(\epsilon)^2 + c_1a_1(\epsilon)^2\sigma(\epsilon) + c_1^2\sigma(\epsilon)^2 \\
\mathbb{E}_{\mathbb{P}'}[X(\epsilon)^2] &= \mathbb{E}_{\mathbb{P}'}[(P_1^{(\epsilon)}(X(\epsilon)))^2] \\
&= c_1\sigma(\epsilon).
\end{aligned}$$

If there exists a constant $c > 0$ such that $\|X(\epsilon)\|_{L^4(\Omega';\mathbb{R})} \le c\|X(\epsilon)\|_{L^2(\Omega';\mathbb{R})}$ for all $\epsilon > 0$, then

$$(c_2 + c_1^2) + \frac{c_1a_1(\epsilon)^2}{\sigma(\epsilon)} \le c^2c_1^2$$

for all $\epsilon > 0$. Therefore need $\mathbb{E}[X(\epsilon)^3] = O(\sigma(\epsilon)^{3/2})$.

Definition 11. *For any* $\ell \in \mathbb{N}$ *and* $j \in \{1, \dots, \ell\}$ *the operator* $D_j^* : \mathcal{P}(E \otimes H_0^{\otimes \ell}) \to \mathcal{P}(E \otimes H_0^{\otimes(\ell-1)})$ *is defined by*

$$D_j^* F := \sum_{i_0,\dots,i_\ell=1}^{N} D^*\Big(F_{i_0,\dots,i_\ell} \dot{h}_{i_j}(\cdot,\cdot)\Big) e_{i_0} \otimes_{a\in\{1,\dots,\ell\}\setminus\{j\}} h_{i_a}$$

for any

$$F = \sum_{i_0,\dots,i_\ell=1}^{N} F_{i_0,\dots,i_\ell} \eta_{i_0} \otimes_{a\in\{1,\dots,\ell\}} h_{i_a} \in \mathcal{P}(E \otimes H_0^{\otimes \ell})$$

where $\{F_{i_0,\dots,i_\ell}\}_{i_0,\dots,i_\ell=1}^{N} \subset \mathcal{P}(\mathbb{R})$, $\{h_i\}_{i\in\{1,\dots,N\}} \subset H_0$ *and* $N \in \mathbb{N}$.

As generally $\nu(\mathbb{R}_0^d) = \infty$, combinations of D and partial $D_\cdot^*$ operators need each partial $D_\cdot^*$ operator to apply to an $F \in \mathcal{P}(E \otimes H_0^{\otimes \ell})$ for non-zero ℓ.

For any $\ell, k \in \mathbb{N} \cup \{0\}$ define

$$\mathcal{A}_{\ell,k} := \Big\{ (i_1, \ldots, i_{2k}) \in \{0,1\}^{2k} \Big| \sum_{j=1}^{2k} i_j = k, \quad \ell + \sum_{j=0}^{u} i_{2k-2j} \geq \sum_{j=0}^{u} i_{2k-2j-1} \quad \forall u \in \{0, \ldots, k-1\} \Big\}$$

and for each $i := (i_1, \ldots, i_{2k}) \in \mathcal{A}_{\ell,k}$ denote by $|i|$ the value $|i| := \sum_{j=1}^{k} i_{2j} - \sum_{j=1}^{k} i_{2j-1}$.

For any $\ell, k \in \mathbb{N} \cup \{0\}$ and $i := (i_1, \ldots, i_{2k}) \in \mathcal{A}_{\ell,k}$ define

$$\mathcal{D}_i := \Bigg\{ (j_1, \ldots, j_k) \in \Big\{ 0, 1, \ldots, \ell + \sum_{m=1}^{k} i_{2m} \Big\}^k \Bigg| \\ j_m \in \left\{ \begin{array}{cc} \{1, \ldots, \ell + \sum_{a=m}^{k} i_{2a}\} \backslash \{j_{m+1}, \ldots, j_k\} & \text{if } i_{2m-1} \neq 0, \\ \{0\} & \text{if } i_{2m-1} = 0. \end{array} \right\} $$

For any $\ell, k \in \mathbb{N} \cup \{0\}$, $i = (i_1, \ldots, i_{2k}) \in \mathcal{A}_{\ell,k}$ and $j = (j_1, \ldots, j_k) \in \mathcal{D}_i$ then define the operator $D^i_j : \mathcal{P}(E \otimes H_0^{\otimes \ell}) \to \mathcal{P}(E \otimes H_0^{\otimes(\ell+|i|)})$ by

$$D^i_j := (D^*_{j_1})^{i_1} \circ D^{i_2} \circ \cdots \circ (D^*_{j_k})^{i_{2k-1}} \circ D^{i_{2k}}$$

where D^0 and $(D^*_0)^0$ are both the identity map.

Lemma 12. *For any* $p \in (1, \infty)$, $\ell, k \in \mathbb{N} \cup \{0\}$, $i \in \mathcal{A}_{\ell,k}$ *and* $j \in \mathcal{D}_i$ *then* $D^i_j : \mathcal{P}(E \otimes H_0^{\otimes \ell}) \to \mathcal{P}(E \otimes H_0^{\otimes(\ell+|i|)})$ *has a closed extension* $D^i_j : \mathrm{Dom}_{L^p(\Omega; E \otimes H^{\otimes \ell})}(D^i_j) \to L^p(\Omega; E \otimes H^{\otimes(\ell+|i|)})$.

Definition 13. *For any* $\ell \in \mathbb{N} \cup \{0\}$ *the family of norms* $\{\| \cdot \|_{\mathbb{D}^{p,n}(E \otimes H^{\otimes \ell})} | p \in (1, \infty), n \in \mathbb{N} \cup \{0\}\}$ *on* $\mathcal{P}(E \otimes H_0^{\otimes \ell})$ *is defined by*

$$\| \cdot \|_{\mathbb{D}^{p,n}(E \otimes H^{\otimes \ell})} = \sum_{k=0}^{n} \sum_{i \in \mathcal{A}_{\ell,k}} \sum_{j \in \mathcal{D}_i} \| D^i_j \cdot \|_{L^p(\Omega; E \otimes H^{\otimes(\ell+|i|)})}$$

for each $p \in (1, \infty)$ *and* $n \in \mathbb{N} \cup \{0\}$. *Define the family of Sobolev spaces* $\{\mathbb{D}^{p,n}(E \otimes H^{\otimes \ell}) | p \in (1, \infty), n \in \mathbb{N} \cup \{0\}\}$ *such that for each* $p \in (1, \infty)$ *and* $n \in \mathbb{N} \cup \{0\}$ *then* $\mathbb{D}^{p,n}(E \otimes H^{\otimes \ell})$ *is the Banach space completion of* $\mathcal{P}(E \otimes H_0^{\otimes \ell})$ *with respect to the norm* $\| \cdot \|_{\mathbb{D}^{p,n}(E \otimes H^{\otimes \ell})}$.

Lemma 14. *For each fixed $\ell \in \mathbb{N} \cup \{0\}$, the family of norms* $\{\|\cdot\|_{\mathbb{D}^{p,n}(E\otimes H^{\otimes\ell})} | p \in (1,\infty), n \in \mathbb{N} \cup \{0\}\}$ *are compatible.*

Definition 15. *For any $\ell \in \mathbb{N} \cup \{0\}$ the class of $E \otimes H^{\otimes\ell}$-valued smooth pure jump Lévy functionals (SPJLFs) $\mathbb{D}^{\infty}(E \otimes H^{\otimes\ell})$ is defined by*

$$\mathbb{D}^{\infty}(E \otimes H^{\otimes\ell}) := \bigcap_{p\in(1,\infty)} \bigcap_{n\in\mathbb{N}\cup\{0\}} \mathbb{D}^{p,n}(E \otimes H^{\otimes\ell}).$$

For any $\ell, k \in \mathbb{N} \cup \{0\}$, $i \in \mathcal{A}_{\ell,k}$ and $j \in \mathcal{D}_i$ then given any continuous linear functional $Y \in L^p(\Omega; E \otimes H^{\otimes(\ell+|i|)})^*$ we denote by $(D^i_j)^* Y$ the continuous linear functional in $\mathrm{Dom}_{L^p(\Omega; E\otimes H^{\otimes \ell})}(D^i_j)^*$ defined by

$$\langle (D^i_j)^* Y, F \rangle := \langle Y, D^i_j F \rangle$$

for all $F \in \mathrm{Dom}_{L^p(\Omega; E\otimes H^{\otimes \ell})}(D^i_j)$.

Proposition 16. *For all* $p \in (1, \infty)$ *and* $\ell, n \in \mathbb{N} \cup \{0\}$ *the dual space* $\mathbb{D}^{p,n}(E \otimes H^{\otimes \ell})^*$ *can be represented as*

$$\mathbb{D}^{p,n}(E \otimes H^{\otimes \ell})^* = \Big\{ Y : \mathbb{D}^{p,n}(E \otimes H^{\otimes \ell}) \to \mathbb{R} \Big| \\ Y = \sum_{k=0}^{n} \sum_{i \in \mathcal{A}_{\ell,k}} \sum_{j \in \mathcal{D}_i} (D^i_j)^* Y_{i,j} \text{ where } Y_{i,j} \in L^p(\Omega; E \otimes H^{\otimes(\ell+|i|)})^* \Big\}$$

References

[Gun86] R. F. Gundy. Sur les transformations de Riesz pour le semi-groupe d'Ornstein-Uhlenbeck. C. R. Acad. Sc. Paris, t. 303, Série I, n. 19:967-970, 1986.

[Itô56] K. Itô. Spectral Type of the Shift Transformation of Differential Processes With Stationary Increments. Transactions of the American Mathematical Society, 81(2):253-263, 1956.

[LLM80] É. Lenglart, D. Lépingle, and M. Pratelli. Présentation unifiée de certaines inégalités de la théorie des martingales. Séminaire de probabilités (Strasbourg), tome 14:26-48, 1980.

[Løk04] A. Løkka. Martingale Representation of Functionals of Lévy Processes. Stochastic Analysis and Applications, 22(4):867-892, 2004.

[Mey84] P.-A. Meyer. Transformations de Riesz pour les lois gaussiennes. Séminaire de probabilités (Strasbourg), tome 18:179-193, 1984.

[Pic96] J. Picard. On the existence of smooth densities for jump processes. Probability Theory and Related Fields, 105:481-511, 1996.

[Pis88] G. Pisier. Riesz transformations: a simpler analytic proof of P.A. Meyer's inequality. Séminaire de probabilités (Strasbourg), tome 22:485-501, 1988.

[Sur83] D. Surgailis. On Poisson multiple stochastic integrals and associated equilibrium Markov processes. Lecture Notes in Control and Information Sciences, 49:233-248.

[Sur84] D. Surgailis. On multiple Poisson stochastic integrals and associated Markov semigroups. Probability and Mathematical Statistics, 3:217-239, 1984.

[Wat84] S. Watanabe. Stochastic Differential Equations and Malliavin Calculus, volume 73 of Tata Institute of Fundamental Research, Lectures on Mathematics and Physics. Springer-Verlag, 1984.

[Wat87] S. Watanabe. Analysis of Wiener functionals (Malliavin calculus) and its applications to heat kernels. The Annals of Probability, 15:1-39, 1987.