Convergence in variation for the laws of Poisson functionals under weak regularity assumptions

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Main Topics.

1. Absolute continuity and convergence in variation criteria for the laws of Levy functionals, based on the Davydov's stratification method
2. Ergodic properties of jump diffusions

These topics are mainly based on

A.M.Kulik, Absolute continuity and convergence in variation of distributions for functionals of Poisson point measure. (2008); accepted to *Jour. Theor. Prob.*

A.M.Kulik, Exponential ergodicity of the solutions to SDE's with a jump noise. *Stoch. Proc. Appl.* **119**, 602 – 632 (2009).

Davydov's stratification method: an abstract framework.

Let $\{T_h, h \in \mathbb{R}^m\}$ be a *group of transformations* of a probability space Ω. Consider equivalence classes w.r.t. the relation

$$\omega \sim \omega' \Leftrightarrow \exists h : T_h\omega = \omega'.$$

Denote an equivalence class by γ, and the set of classes by Γ.
Then Ω is "stratified"; that is, represented as the collection of "layers" or "orbits"

$$\Omega = \bigsqcup_{\gamma\in\Gamma} \gamma$$

$$\omega \simeq (\gamma, h), \quad \Omega \simeq \Gamma \times \mathbb{R}^m.$$

$$P(A) = \int_\Gamma P_\gamma(A_\gamma)\pi(d\gamma), \quad A_\gamma = \{h : (\gamma, h) \in A\}.$$

Davydov's stratification method: an abstract framework.

The problem of studying of absolute continuity of the law of $f:\Omega\to\mathbb{R}$ w.r.t. P is reduced to the family of similar problems on every layer γ:

$$P\circ f^{-1}\ll\lambda^m \text{ provided } P_\gamma\circ f_\gamma^{-1}\ll\lambda^m \quad \text{for } \pi-\text{a.a. } \gamma\in\Gamma.$$

Assume the transformations $\{T_h\}$ to be *admissible* for the basic probability P; that is, $P\circ T_h^{-1}\sim P, h\in\mathbb{R}^m$. Then

$$P_\gamma\ll\lambda^m \quad \text{for } \pi-\text{a.a. } \gamma\in\Gamma$$

A. Skorokhod (1975).
Change-of-variables formula: if $\mu(dx)=p_\mu(x)\,dx$ and $F:\mathbb{R}^m\to\mathbb{R}^m$ has a.s. non-degenerate derivative, then the image measure $\mu\circ F^{-1}$ has the density

$$p_{\mu\circ F^{-1}}(x)=\sum_{y:F(y)=x}\frac{1}{|\det F'(y)|}p_\mu(y).$$

$$f(T_h\omega)=f(\gamma,t+h),\quad \omega=(\gamma,t)$$

Davydov's stratification method: an abstract framework.

Define the derivative w.r.t. $\{T_h\}$ by

$$(Df, h) = \lim_{\varepsilon \to 0} \frac{f \circ T_{\varepsilon h} - f}{\varepsilon}, \quad h \in \mathbb{R}^m. \tag{1.1}$$

Theorem 1.1

For an $\mathbb{R}^m$-valued function $f = (f_1, \dots, f_m)$, denote

$$\sigma_f = \Big((Df_j, Df_k) \Big)_{j,k=1}^m .$$

Then the image measure of μ, restricted to $\mathcal{N}(f) = \{\det \sigma_f \neq 0\}$, under f is absolutely continuous.
Yu.Davydov (1978), Yu.Davydov, M.Lifshits, N.Smorodina (1995).

L_p derivative: (1.1) holds true in L_p sence; equivalent to the Sobolev derivative defined by a closure procedure (S. Albeverio, M. Röckner (1990)).

Weakening regularity assumptions: Lusin-type theorem for differentiable mappings.

Proposition 1.1.

Let function $f : \mathbb{R}^m \to \mathbb{R}$ be *approximatively differentiable* w.r.t. every coordinate on a set $A \subset \mathbb{R}^m$ (e.g. differentiable a.s.). Then for every $\varepsilon > 0$ there exists a function $f_\varepsilon \in C^1(\mathbb{R}^m, \mathbb{R})$ such that

$$\mathrm{Leb}(x \in A : f_\varepsilon(x) \neq f(x) \text{ or } f'_\varepsilon(x) \neq f'(x)) < \varepsilon.$$

G.Federer (1969).

A.s. derivative: (1.1) holds true almost surely. A.Pilipenko (1997).

Sufficient condition for convergence in variation

Theorem 1.2

Consider a sequence of $\mathbb{R}^m$-valued random vectors $\{f^n, n \geq 1\}$ such that, for a given grid $\mathcal{G}$ of dimension m, every component $f^n_i, i = 1, \dots, m$ of the vector f^n is a.s. differentiable. Suppose that

$$f^n_j \to f_j, \quad D^{\mathcal{G}}_i f^n_j \to D^{\mathcal{G}}_i f_j \text{ in probability}, \quad n \to +\infty, \quad i, j = 1, \dots, m.$$

Suppose additionally that $\{f^n\}$ has **uniformly dominated increments** on a set Ω'. Then, for every $A \subset \mathcal{N}(f, \mathcal{G}) \cap \Omega'$,

$$P\Big|_A \circ f_n^{-1} \to P\Big|_A \circ f^{-1}, \quad n \to +\infty$$

in variation.

Convergence in variation

Definition

The sequence of the measurable functions $\{f_n : \Omega \to \mathbb{R}, n \geq 1\}$ is said to have *uniformly dominated increments* on a set Ω', if there exist a random variable ϱ and a family of jointly measurable functions $\{g_i : \Omega \times \mathbb{R} \to \mathbb{R}\}$ such that
(i) for every i and almost every ω, the function $g_i(\omega, \cdot)$ is an increasing one;
(ii) $\varrho > 0$ almost surely and, for every $n \geq 1, \omega \in \Omega'$,

$$|T_t f_n(\omega) - T_s f_n(\omega)| \leq \sum_{i=1}^{m} \Big[g_i(\omega, t_i \vee s_i) - g_i(\omega, t_i \wedge s_i) \Big],$$

$$\|t\|, \|s\| < \varrho(\omega), T_t\omega \in \Omega', \ T_s\omega \in \Omega'.$$

D.Alexandrova, V.Bogachev, A.Pilipenko (1999): a criterion for convergence in variations of image measures in $\mathbb{R}^m$.
A.Kulik (2005): an extension for *uniformly approximatively Lipschitz* sequences. Loosely speaking, the Lusin-type theorem should hold true uniformly; that is, with one small exlusive set for all functions from the sequence. This is provided by an assumption that the increments of these functions are dominated by increments of a finite set of auxiliary functions.

Admissible perturbations on a Lévy space.

The Lévy probability space is the space of realizations for a Poisson point measure ν with the intensity measure

$$\mu(dt, du) = dt\mu(du);$$

μ is the **Lévy measure**.
Because of smooth Lebesgue component dt, perturbations

$$(\tau, u) \mapsto (\tau + \theta(\tau, u), u)$$

of ν considered as a collection of "jumps" (τ, u) are admissible.

E.Carlen, E.Pardoux (1990), T.Elliott, A.Tsoi (1993). **A group** of admissible transformations: A.Kulik (1999).

Time-stretching transformations

Denote

$$H = L_\infty(\mathbb{R}^+) \cap L_2(\mathbb{R}^+), \quad Jh(\cdot) = \int_0^{\cdot} h(s)\, ds, h \in H.$$

For $h \in H$, consider the Cauchy problem

$$z'_{x,h}(s) = Jh(z_{x,h}(s)), \quad s \in \mathbb{R}, \quad z_{x,h}(0) = x$$

and define the transformation S_h of the time axis by

$$S_h x = z_{x,h}(1).$$

Heuristically, under S_h every infinitesimal segment dx of $\mathbb{R}^+$ is stretched in $e^{h(x)}$ times. Denote $\mu_{fin} = \{\Gamma \in \mathcal{B}(\mathbb{R}^d) : \mu(\Gamma) < +\infty)\}$ and for $\Gamma \in \mu_{fin}$ define the transformation T_h^Γ of the configuration of a point measure ν by the rule:

$$T_h^\Gamma : (\tau, u) \rightsquigarrow \begin{cases} (\tau, u), & u \notin \Gamma, \\ (S_h\tau, u), & u \in \Gamma. \end{cases}$$

The family $\{T_{th}^\Gamma, t \in \mathbb{R}\}$ is a group of admissible transformations for ν.

Differential grids and multi-dimensional groups of transformations

For various h, g transformations S_h, S_g **not necessarily commute**. This motivates the following

Definition

A family $\mathcal{G} = \{[a_i, b_i) \subset \mathbb{R}^+, h_i \in H_0, \Gamma_i \in \mu_{fin}, i \leq m\}$ is called *a differential grid* if
(i) for every $i \neq j$, $\Big([a_i, b_i) \times \Gamma_i\Big) \cap \Big([a_j, b_j) \times \Gamma_j\Big) = \varnothing$;
(ii) for every $i \leq m$, $Jh_i > 0$ inside (a_i, b_i) and $Jh_i = 0$ outside (a_i, b_i).

Denote $T_t^i = T_{th_i}^{\Gamma_i}$, $t \in \mathbb{R}, i = 1, \dots, m$,

$$T_z^{\mathcal{G}} = T_{z_1}^1 \circ T_{z_2}^2 \circ \cdots \circ T_{z_m}^m, \quad z = (z_1, \dots, z_m) \in \mathbb{R}^m.$$

Proposition 2.1

The family $\{T_z^{\mathcal{G}}, z \in \mathbb{R}^m\}$ is a group of admissible transformations for ν.

Absolute continuity and convergence in variation

Theorem 2.1

Consider a random vector $f=(f_1,\dots,f_m)$ and a grid $\mathcal{G}$ of dimension m. Let every component of the vector f be differentiable w.r.t. $\mathcal{G}$ either in a.s. or in L_p sense.
Denote $\Sigma^{f,\mathcal{G}}=(D_i^{\mathcal{G}}f_j)_{i,j=1}^m$ and put
$\mathcal{N}(f,\mathcal{G})=\{\omega:$ the matrix $\Sigma^{f,\mathcal{G}}(\omega)$ is non-degenerate$\}$. Then

$$P\Big|_{\mathcal{N}(f,\mathcal{G})}\circ f^{-1}\ll\lambda^m.$$

If a sequence $f_n, n\geq 1$ converge to f together with respective derivatives and has uniformly dominated increments, then

$$P\Big|_{\mathcal{N}(f,\mathcal{G})}\circ f_n^{-1}\to P\Big|_{\mathcal{N}(f,\mathcal{G})}\circ f^{-1},\quad n\to+\infty$$

in variation.

Application to Itô-Lévy type SDE's

$$\begin{aligned} X(x,t) = x &+ \int_0^t a(X(x,s))\,ds \\ &+ \int_0^t \int_{|u|>1} c(X(x,s-),u)\nu(ds,du) \\ &+ \int_0^t \int_{|u|\le 1} c(X(x,s-),u)[\nu(ds,du) - ds\mu(du)], \quad t\in\mathbb{R}^+. \end{aligned} \tag{2.1}$$

Differentiability of the solution

Theorem 2.2

For every $i = 1, \ldots, m$, derivative

$$Y_i(x,\cdot) = (D_i^{\mathcal{G}} X_j(x,\cdot))_{j=1}^m$$

is well defined in a.s. sense, and satisfies linear SDE

$$\begin{aligned} Y_i(x,t) = & \int_0^t \int_{\Gamma_i} \Delta(X(x,s-),u) J h_i(s)\,\nu(ds,du) \\ & + \int_0^t [\nabla a](X(x,s)) Y_i(x,s)\,ds \\ & + \int_0^t \int_{\mathbb{R}^d} [\nabla c]_x (X(x,s-),u) Y_i(x,s)\,[\nu(ds,du) - \mathbb{1}_{|u|\le 1} s d\mu(du)], \end{aligned}$$

where

$$\Delta(x,u) = [a(x + c(x,u)) - a(x)] - (\nabla_x c(x,u), a(x)).$$

Differentiability of the solution

The principal assumption is

$$c(x,u) = c_0(u) + c_1(x,u),$$

$$c_0 \wedge 1 \in L_2(\mu), \quad |c_1(x,u)| + |[c_1]'_x(x,u)| \leq c_2(u), \quad c_2 \wedge 1 \in L_1(\mu).$$

Particular cases: A.Kulik (2005), $c = c_1$; I.Nourdin, T.Simon (2006), $c = c_0, m = 1$.

Crucial observation I: if $c_1 \neq 0$, the differentiability holds true in a.s. sense, only.

$$c(x,u) + c(x + c(x,u), v) \neq c(x,v) + c(x + c(x,v), u).$$

Crucial observation II: if $a \equiv 0$ then $\Delta \equiv 0$ and $Y_i \equiv 0$. Non-degeneracy of the derivative requires the drift to be non-trivial.

Itô-Lévy type SDE's: absolute continuity and convergence in variation

Denote

$$\mathcal{E}_s^t = I_{\mathbb{R}^m} + \int_s^t \nabla a(X(x,r))\mathcal{E}_s^r\, dr + \int_0^t \int_{\mathbb{R}^d} [\nabla c]_x (X(x,s-),u)\mathcal{E}_{s-}^r\, \tilde{\nu}(ds,du),$$

$$\mathcal{S}(x,t) = \operatorname{span}\left\{\mathcal{E}_\tau^t \Delta(X(x,\tau-),p(\tau)), \tau \in \mathcal{D}\cap[0,t]\right\}.$$

Theorem 2.3

1. The distribution of $X(x,t)$, restricted to the set $\{\mathcal{S}(x,t)=\mathbb{R}^m\}$, possesses a density.
2. Let $x_n\to x, t_n\to t>0$. Then the distributions of $X(x_n,t_n)$, restricted to the set $\{\mathcal{S}(x,t)=\mathbb{R}^m\}$, converge in variation to the distribution of $X(x,t)$ restricted to the same set.

Explicit sufficient conditions

One set of sufficient conditions

Assume

I. Asymptotic expansion for the jump coefficient: $c(x,u)=\chi(x)u+\delta(x,u)$,

$$\|\delta(x,u)\|+\|\nabla_x\delta(x,u)\|=o(\|u\|),\quad \|u\|\to 0.$$

II. Joint non-degeneracy condition:

$$\operatorname{rank}\Big[\nabla a(x_*)\chi(x_*)-\nabla\chi(x_*)a(x_*)\Big]=m.$$

III. *Cone condition* on the Lévy measure: for every $v\in\mathbb{R}^m, v\neq 0$, there exists $\varrho\in(0,1)$:

$$\mu\Big(u:\langle u,v\rangle\geq\varrho|u|\Big)=+\infty.$$

Then $P(\mathcal{S}(x,t)=\mathbb{R}^m)=1$ and consequently there exists $p_t(x,y)$.

Cone condition is similar to the *Yamazato condition*: for every proper subspace L,

$$\mu(\mathbb{R}^m\setminus L)=+\infty.$$

Quantitative erqodicity ((r,ϕ,ψ)-ergodicity).

For a probability measure γ on $\mathbb{X}=\mathbb{R}^m$, denote $\gamma_t(dx)=\int_{\mathbb{X}} P(X(t,y)\in dx)\gamma(dy)$. A probability measure π is called *invariant* if $\pi_t=\pi, t>0$.

Definition

The process X is (r,ϕ,ψ)-ergodic if the class of invariant measures for X contains exactly one measure π and

$$\|\gamma_t-\pi\|_{\phi,var}\leq r(t)\int_{\mathbb{X}}\psi\,d\gamma,\quad r(t)\to 0,\quad t\to\infty \tag{3.1}$$

with ϕ*-variation* of a signed measure κ being defined by $\|\kappa\|_{\phi,var}=\int_{\mathbb{X}}\phi\,d[\kappa_++\kappa_-]$.

$\phi=\psi\equiv 1\Rightarrow$ process X is *uniformly ergodic*

Applications: CLT, LDP, statistical inference.

Irreducibility conditions.

Minorization condition; W.Doeblin, 1940

For a given set A, for some $t > 0, \alpha > 0$, and probability measure κ,

$$P_t(x, dy) \geq \alpha\kappa(dy), \quad x \in A. \tag{3.2}$$

Integral condition; R.Dobrushin, 1971

For a given set A, for some $t > 0$

$$\sup_{x,x' \in A} \|P_t(x, \cdot) - P_t(x, \cdot)\|_{var} < 2. \tag{3.3}$$

Irreducibility for jump diffusions

Theorem 3.1

Assume that the process X is topologically irreducible; that is, there exists $x_* \in \mathbb{R}^m$ such that for every $x \in \mathbb{R}^m$ the support of $P_t(x, dy)$ contains x_* for some $t > 0$. Assume additionally that $P(\mathcal{S}(x_*, t) = \mathbb{R}^m) > 0$ for some $t > 0$.
Then the process X satisfies the Dobrushin condition on every compact subset of $\mathbb{R}^m$.

Ergodicity criteria.

Global irreducibility ($A = \mathbb{R}^m$) $\Rightarrow$ uniform ergodicity

Local irreducibility + a *recurrence condition* $\Rightarrow$ (r, ϕ, ψ)-ergodicity.

Recurrence conditions.

By definition, a measurable function $f:\mathbb{X}\to\mathbb{R}$ belongs to the domain of the *extended generator* $\mathcal{A}$ of a Markov process X if there exists a measurable function $g:\mathbb{X}\to\mathbb{R}$ such that the process

$$f(X(t))-\int_0^t g(X(s))\,ds,\quad t\in\mathbb{R}^+$$

is well defined and is an $\mathbb{F}^X$ – martingale w.r.t. to any measure $P_x, x\in\mathbb{X}$. Notation: $\mathcal{A}f=g$.

Lyapunov condition

$$\mathcal{A}\psi\leq -\Phi(\psi)+C\mathbb{I}_A. \tag{3.4}$$

$\Phi:\mathbb{R}^+\to\mathbb{R}^+$ is monotonous, increasing to $+\infty$, $\Phi(0)=0$.

Recurrence for jump diffusions.

By analogy with the diffusive case (A.Veretennikov (1987, 1999); S.Klokov, A. Veretennikov (2003)), assume

$$|c(x,u)| \le C_1|u|, \quad \int_{|u|>1} e^{C_2|u|}\mu(du) < +\infty$$

with some positive $C_{1,2}$. Denote

$$\varrho_\sigma = \limsup_{|x|\to\infty} \left\langle a(x), \frac{x}{|x|^\sigma} \right\rangle, \quad \sigma \ge 0.$$

Recurrence: Lyapunov conditions.

Theorem 3.2

I (*exponential case*). If $\varrho_1 = -\infty$ then (3.4) holds true with

$$\Phi(v) = v, \quad \psi(x) = e^{c\chi(x)}, \quad \chi \in C^2 \text{ and } \chi(x) = |x|, \quad |x| \geq 1.$$

II (*sub-exponential case*). If $\varrho_\sigma = -\infty$ with $\sigma \in (0, 1)$ then (3.4) holds true with

$$\Phi(v) = v|\ln v|^{2-\frac{2}{\sigma}}, \quad \psi(x) = e^{c\chi^\sigma(x)}.$$

III (*polynomial case*). If $\varrho_0 = -\infty$ then (3.4) holds true with

$$\Phi(v) = v^{\frac{r-1}{r}}, \quad \psi(x) = (1 + c|x|^2)^r, \quad r > 1.$$

(r, ϕ, ψ)-ergodicity for jump diffusions.

Theorem 3.3

Under "irreducibility + recurrence" conditions listed above, the process X is (r, ϕ, ψ)-ergodic with

$$\phi(x) = \psi(x) = e^{c_1|x|}, \quad r(t) = c_2 e^{-c_3 t} \quad \text{in the exponential case;}$$

$$\phi(x) = e^{c_1|x|^\sigma}, \quad \psi(x) = e^{c_2|x|^\sigma} \quad r(t) = c_3 e^{-c_4 t^{\frac{\sigma}{2-\sigma}}} \quad \text{in the sub-exponential case;}$$

$$\phi(x) = (1+|x|^2)^{c_1 r}, \psi(x) = (1+|x|^2)^{c_2 r}, r(t) = c_3(1+|x|^2)^{c_4 r} \text{ in the polinomial case.}$$

Constants $c_1 - c_4$ can be expressed explicitly.

Example: Lévy driven non-linear Ornstein-Uhlenbeck process.

$$dX(x,t) = a(X(x,t))\,dt + dZ(t).$$

Proposition 3.1

Assume that the Lévy measure μ of the process Z is non-zero and for some $c > 0$ either

$$\limsup_{|x|\to+\infty} a(x)\operatorname{sign} x < 0, \quad \int_{|u|>1} e^{c|u|}\mu(du) < +\infty \quad \text{or}$$

$$\limsup_{|x|\to+\infty} \frac{a(x)}{x} < 0, \quad \int_{|u|>1} |u|^c\mu(du) < +\infty$$

Then X is $(Ce^{-\beta t}, \phi, \phi)$-ergodic.

The whole approach appears to be well adjusted with the matter of the problem. On one hand, convergence in variation on a non-trivial part of the probability space is exactly what one needs to prove the Dobrushin condition. On the other hand, the claim for the drift coefficient a to be non-degenerate is non-restrictive because the process anyway should be supposed to be recurrent.

Itô-Lévy type SDE's: irregularity of the density

Proposition 2.2

Assume

$$\liminf_{\varepsilon\to 0}\left[\varepsilon^2 \ln \varepsilon^{-1}\right]^{-1}\int_{\mathbb{R}}(u^2\wedge\varepsilon^2)\mu(du)=0. \tag{4.1}$$

Then the transition probability density p_t, if exists, does not belong to any $L_{p,loc}(\mathbb{R}^m)$.

Heuristically, for a Lévy noise to produce a regular transition probability density, it is **necessary** that "a lot of small jumps" are available. If the Lévy measure is not "massive near the origin", i.e (4.1) holds, then the density provided by Theorem 2.4 is highly irregular.

Recall that, under sufficient conditions that provide

$$P(\mathcal{S}(x,t)=\mathbb{R}^m)=1,$$

the function

$$\mathbb{R}^m\ni x\mapsto p_t(x,\cdot)\in L_1(\mathbb{R}^m)$$

is continuous.

An extension: Hörmander-type condition

Hörmander-type generalization of the above conditions are available, A.Kulik (2005). This generalization, applied to the Levy driven Ornstein-Uhlenbeck process

$$dX(t) = AX(t)dt + BdZ(t),$$

leads to the following criterion.

Theorem 2.4

Condition

$$\text{(H2)} \qquad \mathrm{Rank}\,[AB, \dots, A^m B] = m$$

is sufficient for $X(t)$ to have a probability density if the Levy measure of Z satisfies the Yamazato condition.
This condition is also necessary, in a sense.
S.Bodnarchuk, A.Kulik (2008), T.Simon (2009).

An extension: SDE's with variable rate of jumps

Similar results are available for solutions of SDE's of the type

$$X(x,t) = x + \int_0^t a(X(x,s))\,ds + \int_0^t \int_{|u|>1} \int_0^{b(X(x,s-),u)} c(X(x,s-),u,v)\nu(ds,du,dv) + \int_0^t \int_{|u|\leq 1} \int_0^{b(X(x,s-),u)} c(X(x,s-),u)[\nu(ds,du,dv) - ds\mu(du)dv].$$

Assumptions on the coefficients a, b, c are comparable with those from N.Fournier (2008) and V.Bally (2008). Assumptions on the Lévy measure are essentially milder. On the other hand, the statement that the transition probability density **exists and is** L_1**-continuous**, is essentially weaker than C^∞**-differentiability** of the density.