Local Malliavin Calculus for Lévy Processes and Applications

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Motivation

Over the past years, it has been several attempts for extending the Malliavin calculus to a scenario driven by Lévy processes. (Bismut (1983), Leandre (1985), Bichteler, Graveraux and Jacod (1987)).

We can point out two different lines to handle with this situation.

- The papers associated with the addition of an extra jump, which imply that the Malliavin "derivative" is a difference operator.
- The papers associated with the use of a true derivative operator, where we can use the chain rule.

For the standard Poisson process,

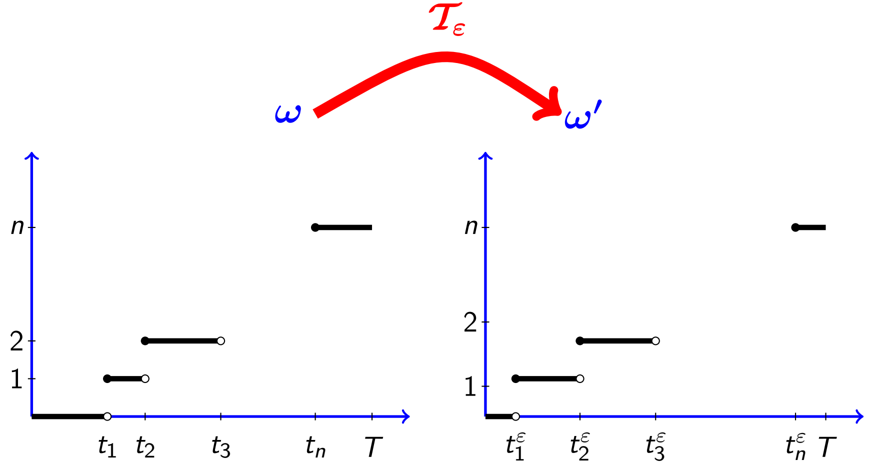
- (difference) Nualart and Vives (1990) proved that the difference operator agrees with the annihilation one.
- (derivative) The first attempt in this direction is the paper of Carlen and Pardoux (1990)

Immediately it appeared the necessity of extend these methods for Lévy processes with a more general Lévy measure.

- (difference) Picard, Ishikawa and Kunita, norwegian school of probabilities, (S-U-V), Applebaum.
- (derivative) Denis, Bouleau, Kulik, Vally, DiNunno, Decreusefont and Savy.

Our aim in this work is to contribute on the second line, and so to have a a chain rule in order to use the usual proof for getting an absolute continuity criterium. Then apply it to different stochastic differential equations driven by Lévy processes.

Carlen-Pardoux approach to Malliavin derivative on the Poisson space. The shift operator.



$\mathcal{T}_\varepsilon : \Omega \longrightarrow \Omega$, $P \circ \mathcal{T}_\varepsilon^{-1} << P$ and N is still a $P \circ \mathcal{T}_\varepsilon^{-1}-$Poisson process

Malliavin derivative

The shift $\mathcal{T}_\varepsilon$ is defined using a centered function $g \in L^2_0(0,T)$.

$$\mathbb{D}^0_g = \{F \in L^2(\Omega) : \text{ exists } L^2(\Omega) - \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}\big(\mathcal{T}_\varepsilon F - F\big)\}.$$

For $F \in \mathbb{D}^0_g$,

$$D_g F := L^2(\Omega) - \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}\big(\mathcal{T}_\varepsilon F - F\big).$$

There is a stochastic process

$$\{D_t F, 0 \le t \le T\}$$

such that

$$\int_0^T D_t F\, g(t)\, dt = D_g F, \ \forall g \in L^2_0(0,T).$$

Leon-Tudor result for the Poisson

The starting point of our paper is the following remark due to Tudor and Leon about the Carlen–Pardoux derivative:
Let N be a Poisson process of parameter 1, and $\widetilde{N_t} = N_t - t$ the compensated Poisson process. Denote by $T_1 < \cdots < T_n, \ldots$ the jump times of N. Let $h \in \mathcal{C}^1\big([0, T]\big)$. Then the random variable

$$Y = \int_0^T h(s)\, d\widetilde{N_s} = \sum_{T_n \leq T} h(T_n) - \int_0^T h(s)\, ds$$

is in the domain of the Malliavin derivative in the Carlen-Pardoux sense and the derivative satisfies the following equality:

$$D_t Y = \int_0^T h'(s)\big(\frac{s}{T} - 1_{(t,T]}(s)\big)\, dN_s.$$

In the next subsection, we extend formally this property to define a Malliavin derivative for a Lévy process.

Extension to Lévy process

Let X be a Lévy process with Lévy-Itô representation:

$$X_t = \gamma t + \sigma W_t + \int\int_{(0,t]\times\{|x|>1\}} x dN(s,x) + \lim_{\epsilon\to 0}\int\int_{(0,t]\times\{\epsilon<|x|\le 1\})} x d\tilde{N}(s,x),$$

where W is a standard Brownian motion, N is the jump measure of the process and $d\tilde{N}(t,x) = dN(t,x) - dt\nu(dx)$. Moreover W and N are independent. The limit is a.s., uniform in t in any bounded interval.

Itô (1956) proved that X determines a a centered independently scattered random measure M on $[0,\infty)\times\mathbb{R}$.
For $E\in\mathcal{B}([0,\infty)\times\mathbb{R})$ such that $\mu(E)<\infty$,

$$M(E)=\sigma\int_{E(0)}dW_t+\lim_{n\to\infty}\int\int_{\{(t,x)\in E':\frac{1}{n}\le|x|\le n\}}xd\tilde{N}(t,x).$$

For $A,B\in\mathcal{B}([0,\infty)\times\mathbb{R})$, with $\mu(A)<\infty$ and $\mu(B)<\infty$, we have

$$\mathbb{E}[M(A)M(B)]=\mu(A\cap B),$$

where

$$\mu(dt,dx)=\lambda(dt)\delta_0(dx)+\lambda(dt)x^2\nu(dx),$$

and λ is the Lebesgue measure in $\mathbb{R}$.

For a function $h \in L^2([0, T] \times \mathbb{R}, \mu)$, we can construct the integral (in the $L^2(\Omega)$ sense)

$$M(h) := \int_{[0,T]\times\mathbb{R}} h(t,x)\, dM(t,x),$$

which is the multiple integral of order 1 defined by Itô. This integral is centered, and for $g, h \in L^2([0, T] \times \mathbb{R}, \mu)$,

$$E[M(h)M(g)] = \int_{[0,T]\times\mathbb{R}} h\, g\, d\mu.$$

We can write this integral as

$$M(h) = \sigma \int_0^T h(t,0) dW_t + \int_{[0,T]\times\mathbb{R}_0} h(t,x) x\, d\tilde{N}(t,x).$$

The local Malliavin derivative

$\mathcal{S}$ denotes the family of all functionals of the form

$$f(M(h_1), \dots, M(h_n)),$$

where

- f in $C_b^\infty(\mathbb{R}^n)$ (f and all its partial derivatives are bounded),
- $h_1, \dots, h_n \in L^2([0,T] \times \mathbb{R}, \mu)$ and for all $x \neq 0$, $h_i(\cdot, x) \in C^1([0,T])$, $i \in \{1, \dots, n\}$, and $\partial h_i \in L^2([0,T] \times \mathbb{R}, \mu)$ where ∂ means the partial derivative with respect to time.

The set $\mathcal{S}$ is called the family of smooth random variables, and it is dense in $L^2(\Omega)$.

We will also consider the family $\mathcal{K}$ of all bounded functions $k : [0,T] \times \mathbb{R}_0 \to \mathbb{R}$ such that they and their partial derivative with respect time are in $L^2([0,T] \times \mathbb{R}_0, \lambda \times \nu) \cap L^1([0,T] \times \mathbb{R}_0, \lambda \times \nu)$.

The derivative operator

Definition

Given $k \in \mathcal{K}$, $F \in \mathcal{S}$, ($F = f(M(h_1), \ldots, M(h_n))$) and $\Lambda \in \mathcal{B}(\mathbb{R})$, we define

$$D_t^{\Lambda,k} F = \sum_{i=1}^{n} (\partial_i f)(M(h_1), \ldots, M(h_n)) D_t^{\Lambda,k} M(h_i), \quad t \in [0, T],$$

with

$$D_t^{\Lambda,k} M(h) = \mathbb{1}_{\Lambda}(0)\, \sigma h(t,0) + \int_0^T \int_{\Lambda \cap \mathbb{R}_0} k(s,y) \partial_s h(s,y) y \left(\frac{s}{T} - \mathbb{1}_{[t,T]}(s)\right) dN(s,y).$$

We call $D_t^{\Lambda,k} F$ the local Malliavin derivative of F.

Remarks

We have the following properties for the derivative operators:

- The integral in the right-hand side of the definition of the derivative, which is an integral with respect to the no compensated Poisson measure, is well-defined due to the function $k \in \mathcal{K}$.

$$D_t^{\Lambda,k} M(h) = \mathbb{1}_\Lambda(0)\,\sigma h(t,0) + \int_0^T \int_{\Lambda\cap\mathbb{R}_0} k(s,y)\partial_s h(s,y) y\Big(\frac{s}{T} - \mathbb{1}_{[t,T]}(s)\Big) dN(s,y).$$

- Also we will see that this family $\mathcal{K}$ allow to consider applications to stochastic differential equations. The function $k(t,x)$ is an essential ingredient of the derivative, and it may change from one application to another. To short the notation, in general we will omit that k in the expression $D_t^{\Lambda,k} F$.

- Observe that if $\Lambda = \{0\}$, then $D^{\Lambda,k}$ is the Malliavin's derivative with respect to the Brownian part of the Lévy process X,
- If $\Lambda = \{x\}$ for some $x \neq 0$ with $\nu(\{x\}) \neq 0$, we obtain

$$D_t^{\{x\}} M(h) = x \int_0^T k(s,x) \partial_s h(s,x) (\frac{s}{T} - \mathbb{1}_{[t,T]}(s)) dN_s^x,$$

where N_s^x is the Poisson process on $[0, T]$ that counts the number of jumps of height x.

- Moreover, if the Lévy process is the standard Poisson process, $(x = 1)$, and we take $k(t,x)$ independent of the time parameter, we obtain

$$D_t^{\{1\},k} M(h) = k(1) \int_0^T \partial_s h(s,1) (\frac{s}{T} - \mathbb{1}_{[t,T]}(s)) dN_s^1.$$

- Let $F \in \mathcal{S}$ such that $F = 0$, a.s. Then $D^{\Lambda} F = 0$, $\lambda \otimes P -$ a.e. That means that the operator $D^{\Lambda,k}$ is well-defined.

In the previous remarks we have used the following Fubini type theorem, whose proof is straightforward:

Theorem

Let $f \in L^2([0,T]^2 \times \mathbb{R}, \lambda \otimes \mu)$. Then all the following integrals are well defined and

$$\int_0^T M(f(t,\cdot))\, dt = M\Big(\int_0^T f(t,\cdot)\, dt\Big), \ \ a.s.$$

Next proposition gives a Malliavin integration by parts formula:

Theorem

Let $F \in \mathcal{S}$, and g be a measurable and bounded function on $[0, T]$. Then

$$E\left(\int_0^T (D_t^{\Lambda,k} F) g(t) dt\right) = E\left[F \mathbb{1}_\Lambda(0) \int_0^T \sigma g(s) dW_s\right]$$
$$+ E\left[F \int_0^T \int_{\Lambda \cap \mathbb{R}_0} \left(g(s) - \frac{1}{T}\int_0^T g(t)dt\right) k(s,y) dN(s,y)\right]$$
$$- E\left[F \left(\int_0^T \int_{\Lambda \cap \mathbb{R}_0} \partial_s k(s,y) \left[\int_0^T g(t)(\frac{s}{T} - \mathbb{1}_{[0,s]}(t))dt\right] dN(s,y)\right)\right].$$

We proof first this formula when $\nu(\Lambda) < \infty$, and take the limit.

The following rules of derivation are well known in Brownian Malliavin calculus; they can be translated to our context with the same proof.

Theorem

*(**Chain rule**). Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function with bounded partial derivatives, and let $F = (F_1, \ldots, F_n)$ a random vector such that $F_j \in \mathrm{Dom}\, D^\Lambda$, for $j = 1, \ldots, n$. Then $f(F) \in \mathrm{Dom}\, D^\Lambda$ and*

$$D^\Lambda\big(f(F)\big) = \sum_{j=1}^{n} \partial_j f(F) D^\Lambda F_j.$$

2. *Let $F, G \in \mathrm{Dom}\, D^\Lambda$ such that $G\, D^\Lambda F, F\, D^\Lambda G \in L^2([0,T] \times \Omega)$. Then $F\,G \in \mathrm{Dom}\, D^\Lambda$ and*

$$D^\Lambda(F\,G) = G\, D^\Lambda F + F\, D^\Lambda G.$$

Applying the integration by parts formula and proceeding as in the book of Nualart, we show the following result, taking in account that the bounded functions are dense in $L^2([0,T])$.

Corollary

The operator $D^{\Lambda,k}$ is an unbounded densely defined and closable operator from $L^2(\Omega)$ into $L^2([0,T]\times\Omega)$.

In particular, we have that the operator D^{Λ} has a closed extension, which is also written by D^{Λ}. The domain of this operator is denoted by Dom D^{Λ}.

Absolute continuity criterium

The chain rule allows to prove a criterion for the absolutely continuity of a functional F in a similar way that in the Brownian case.

Theorem

Let $\Lambda \in \mathcal{B}(\mathbb{R})$, $k \in \mathcal{K}$ and $F \in \mathrm{Dom}\, D^{\Lambda}$ such that

$$\int_0^T (D_t^{\Lambda} F)^2 dt > 0$$

a.s. on a measurable set $A \in \mathcal{F}$. Then, $P \circ F^{-1}$ is absolutely continuous on A (i.e., $\lambda(B) = 0$ implies that $P(\{F \in B\} \cap A) = 0$).

It is worth to remark that the criterion is true for every set Λ and weight function $k \in \mathcal{K}$. This is very interesting for applications because we can choose an appropriate Λ and k depending of F.

Derivatives of iterated integrals.

We want to study the derivatives of iterated integrals over sets with finite Lévy measure. Such integrals can be computed pathwise, and its properties can be proved by combinatorial methods.
Let $\Theta \in \mathcal{B}(\mathbb{R})$ be a bounded set such that $0 \notin \overline{\Theta}$ (closure of Θ); then $\nu(\Theta) < \infty$. Write the Poisson random measure

$$N^{\Theta}(B) = \#\{t : \ (t, \Delta X_t) \in B \text{ and } \Delta X_t \in \Theta\}, \quad B \in \mathcal{B}\big((0,\infty) \times \mathbb{R}_0\big).$$

that has intensity $\lambda \otimes \nu_{\Theta}$, where $\nu_{\Theta}(C) = \nu(\Theta \cap C)$, for $C \in \mathcal{B}(\mathbb{R})$.

Define

$$N_t^{\Theta} = N^{\Theta}\big([0,t] \times \Theta\big) < \infty, \text{ a.s.}, \quad t \in [0,T]$$

Note that we can order the jumps in the interval $[0,T]$, $T_1 < \cdots < T_{N_T^{\Theta}}$, Multiple integrals can be considered: Denote by $S_n(\Theta)$ the *simplex*

$$\left\{(t_n, x_1; \ldots ; t_n, x_n) \in ([0, T] \times \mathbb{R}_0)^n : \; t_1 < \cdots < t_n\right\}.$$

For $\phi : S_n(\Theta) \to \mathbb{R}$, define

$$\begin{aligned} J_n^{\Theta}(\phi) &= \int \cdots \int_{S_n(\Theta)} \phi(t_1, x_1; \ldots ; t_n, x_n)\, dN^{\Theta}(t_1, x_1) \cdots dN^{\Theta}(t_n, x_n) \\ &= \sum_{1 \le i_1 < \cdots < i_n \le N_T^{\Theta}} \phi(T_{i_1}, \Delta X_{T_{i_1}}; \ldots ; T_{i_n}, \Delta X_{T_{i_n}}). \end{aligned}$$

with the convention that the sum is zero if $N_T^{\Theta} < n$.

Theorem

For every $p \geq 1$, if $\phi \in L^p\big(S_n(\Theta), (\lambda \otimes \nu_{\Theta})^n\big)$, then $J_n^{\Theta}(\phi) \in L^p(\Omega)$, and

$$\mathbb{E}\left[\left|J_n^{\Theta}(\phi)\right|^p\right] \leq C_{p,n} \int_{S_n(\Theta)} |\phi(t_1, x_1; \ldots ; t_n, x_n)|^p \, dt_1, \cdots dt_n d\nu(x_1) \cdots d\nu(x_n),$$

where the constant $C_{n,p}$ depends only on n and p.

The product formula (see Lee and Shih and Tudor), for this particular case, is simpler and the proof straightforward:

Theorem

Consider $\phi_n : S_n(\Theta) \to \mathbb{R}$ *and* $\phi_1 : [0, T] \times \mathbb{R}_0 \to \mathbb{R}$. *Then,*

$$J_n^{\Theta}(\phi_n)\, J_1^{\Theta}(\phi_1) = J_{n+1}^{\Theta}(\phi_n \widetilde{\otimes} \phi_1) + J_n^{\Theta}(\phi_n * \phi_1),$$

where

$$\phi_n \widetilde{\otimes} \phi_1(t_1, x_1; \dots; t_{n+1}, x_{n+1}) = \sum_{j=1}^{n+1} \phi_n(t_1, x_1; \dots; \widetilde{t_j, x_j}; \dots; t_{n+1}, x_{n+1}) \phi_1(t_j, x_j),$$

and $\widetilde{t_j, x_j}$ *means that this pair is missing, and*

$$\phi_n * \phi_1(t_1, x_1; \dots; t_n, x_n) = \sum_{j=1}^{n} \phi_n(t_1, x_1; \dots; t_n, x_n) \phi_1(t_j, x_j).$$

Lemma

Let $\Theta \in \mathcal{B}(\mathbb{R})$ *be a bounded set such that* $0 \notin \overline{\Theta}$, *and* $\Lambda \in \mathcal{B}(\mathbb{R})$ *with* $\Theta \subset \Lambda$. *Consider* $f \in \mathcal{C}_b^\infty(\mathbb{R})$. *Then* $f(N_T^\Theta) \in \mathrm{Dom}\, D^\Lambda$ *and*

$$D^\Lambda f(N_T^\Theta) = 0.$$

Theorem

Let Θ *and* Λ *as before, and* $\phi \in L^2\big(S_n(\Theta), (\lambda \otimes \nu)^n\big)$ *such that for every* $(x_1, \ldots, x_n) \in \Theta^n$, $\phi(\cdot, x_1; \ldots; \cdot, x_n) \in \mathcal{C}^\infty\big(\overline{S_n}\big)$, *where* $\overline{S_n} = \{0 \le t_1 \le \cdots \le t_n \le T\}$. *Then* $J_n^\Theta(\phi) \in \mathrm{Dom}\, D^\Lambda$ *and*

$$D_t^\Lambda J_n^\Theta(\phi) = \sum_{j=1}^n J_n^\Theta\Big(k(s_j, x_j)\partial_{s_j}\phi(s_1, x_1; \ldots; s_n, x_n)\big(\frac{s_j}{T} - \mathbb{1}_{(t,T]}(s_j)\big)\Big).$$

Corollary

With the previous notations,

$$\phi(T_1, \Delta X_{T_1}; \ldots, T_n, \Delta X_{T_n}) \mathbb{1}_{\{N_T^{\ominus} \geq n\}} \in \mathrm{Dom}\, D^{\Lambda}$$

and, over $\{N_T^{\ominus} \geq n\}$,

$$D_t^{\Lambda} \phi(T_1, \Delta X_{T_1}; \ldots; T_n, \Delta X_{T_n}) = \sum_{j=1}^{n} k(T_j, \Delta X_{T_j}) \partial_j \phi(T_1, \Delta X_{T_1}; \ldots, T_n, \Delta X_{T_n}) \Big(\frac{T_j}{T} - \mathbb{1}_{[0, T_j]}(t) \Big).$$

Note that this result yields that $D^{\{1\}}$, with $k = 1$, agrees with the operator introduced in Carlen and Pardoux for the Poisson process. Also, in general, with the previous notations, the jump time T_j is in the domain of D^{Λ} and

$$D_t^{\Lambda} T_j = k(T_j, \Delta X_{T_j}) \left(\frac{T_j}{T} - \mathbb{1}_{[0, T_j]}(t) \right).$$

Absolutely continuity for stochastic differential equations

In this section we utilize our previous results to find conditions that guarantee that the solution at time T of some stochastic differential equations driven by a Lévy process has density. The key point is that we can choose the convenient set Λ and weight $k(t,x)$ for each type of equation.

We will study the following classes of SDE,

- An equation driven by a Lévy process with continuous part.
- A pure discontinuous equation with a monotone drift.
  - Case with finite Lévy measure.
  - Case with infinite Lévy measure.
- A pure discontinuous equation with no zero Wronskian.

An equation driven by a Lévy process with continuous part

We here consider the solution of the stochastic differential equation with an additive jump noise and a Wiener stochastic integral of the form

$$Z_t = x_0 + \int_0^t b(Z_s)ds + \int_0^t \sigma(Z_s)dW_s + \int_0^t \int_{\mathbb{R}_0} l(y)ydN(s,y), \quad t \in [0,T],$$

where $l \in \mathcal{K}$ and the coefficients b, σ are differentiable on $\mathbb{R}$ with bounded derivatives.
Then, choosing $\Lambda = \{0\}$ and $k \in \mathcal{K}$, and proceeding as in the book of Nualart, we can see that

$$D_t^{\{0\},k} Z_T = \sigma(Z_t) \exp\left(\int_0^t \sigma'(Z_s)dW_s + \int_0^t \left\{ b'(Z_s) - \frac{1}{2}(\sigma'(Z_s))^2 \right\} ds \right),$$

Thus we can state the following Theorem, where we give a set in which the integral of the square of the last derivative is positive.

Theorem

The random variable Z_T is absolutely continuous with respect to the Lebesgue measure on the set $\{S < T\}$, where

$$S = \inf \left\{ t \in [0, T] : \int_0^t 1\!\!1_{\{\sigma(Z_s) \neq 0\}} ds > 0 \right\} \wedge T.$$

Remark We can see that last theorem also holds in the case that the coefficients are only Lipschitz functions with linear growth.

A pure discontinuous equation with a monotone drift.

In this section we consider the following equation with an additive jump noise

$$Z_t = x + \int_0^t f(Z_s)ds + \int_0^t \int_{\mathbb{R}_0} h(y)dN(s,y), \quad t \in [0,T].$$

where $x \in \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$ is a function with a bounded derivative and $h \in L^2(\mathbb{R}_0,\nu) \cap L^1(\mathbb{R}_0,\nu)$. It is well-known that this equation has a unique square-integrable solution.

In order to calculate $D^{\mathbb{R}_0,k} Z_T$, with $k \in \mathcal{K}$, we first consider the equation restricted to jumps of the driven process with jumps size in

$$\Theta_m = \{x \in \mathbb{R} : \ 1/m < |x| < m\},$$

Theorem

Let $\Theta_m = \{x \in \mathbb{R} : \ 1/m < |x| < m\}$, *and*

$$Z_t^{(m)} = x + \int_0^t f(Z_s^{(m)})ds + \int_{[0,t]\times\Theta_m} h(y)dN(s,y), \quad t \in [0,T].$$

Then, as $m \to \infty$, $Z_t^{(m)}$ *converges to* Z_t *in* $L^2(\Omega)$, *for every* $t \in [0,T]$. *That convergence is also a.s. for every* $t \in [0,T]$ *a.e.*

Now we consider the flow $\{\Phi_t(s,x) : 0 \le s \le t \le T \text{ and } x \in \mathbb{R}\}$ associated with the equation . That is $\Phi_t(s,x)$ is the unique solution to the equation

$$\Phi_t(s,x) = x + \int_s^t f(\Phi_u(s,x))du, \quad t \in [s,T].$$

Denote by $X = \{X_t, t \geq 0\}$ the Lévy process associated to the Poisson measure N (obviously with $\gamma = \sigma = 0$).
The processes X and Z jump at the same times, and the jumps height of the solution process Z in a jump time τ is $h(\Delta X_\tau)$.
Denote by $T_1 < T_2 < ..$ the jump times of N^{Θ_m} (the dependence of T_j on m is suppressed to short the notations). The solution of the approximative equation in $\{N_T^{\Theta_m} = n\}$ is given by

$$
\begin{aligned}
Z_t^{(m)} &= \Phi_t(0, x), \quad t \in [0, T_1), \\
Z_{T_1}^{(m)} &= \Phi_{T_1}(0, x) + h(\Delta X_{T_1}), \\
Z_t^{(m)} &= \Phi_t(T_1, Z_{T_1}^{(m)}), \quad t \in [T_1, T_2),
\end{aligned}
$$

and so on.

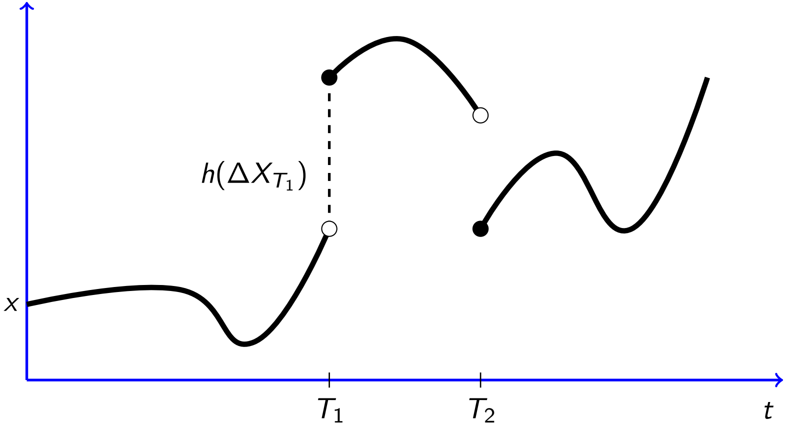


Figure: A trajectory of $Z_t^{(m)}$.

Using induction in n, we have

$$
\begin{aligned}
& D_t^{\mathbb{R}_0,k}\left(Z_T^{\Lambda_m}\mathbb{1}_{[N_T^{\Lambda_m}=n]}\right) \\
= \quad & \mathbb{1}_{[N_T^{\Lambda_m}=n]}\int_0^T\int_{\Lambda_m}\exp\left(\int_s^T f'(Z_u^{\Lambda_m})du\right)\left(f(Z_{s-}^{\Lambda_m})-f(Z_s^{\Lambda_m})\right) \\
& \qquad k(s,y)\left(\frac{s}{T}-\mathbb{1}_{[0,s]}(t)\right)dN(s,y).
\end{aligned}
$$

Taking the limit when m goes to ∞ and, from the dominated convergence theorem, we obtain:

Theorem

$$
\begin{aligned}
D_t^{\mathbb{R}_0,k}Z_T = \int_0^T\int_{\mathbb{R}_0}\exp\left(\int_s^T f'(Z_u)du\right)(f(Z_{s-})-f(Z_s)) \\
k(s,y)\left(\frac{s}{T}-\mathbb{1}_{[0,s]}(t)\right)dN(s,y).
\end{aligned}
$$

Case with finite Lévy measure.

In this section we analyze that the existence of a density for Z_T in the case that the Lévy measure is finite and the drift is a monotone function.

Theorem

Assume that $\nu(\mathbb{R}_0) < \infty$, $h(y) \neq 0$ for $y \in \mathbb{R}_0$ and that f is a monotone function. Then Z_T is absolutely continuous on the set $\left[N_T^{\mathbb{R}_0} \geq 1\right]$.

Proof: We first assume that f is an increasing function. We choose for k the function $(-h \wedge 1) \vee (-1)$ that depends only on y. The function $k(y)$ is bounded and $k(y)$ has the opposite sign than $h(y)$. Then, $(f(Z_{s-}) - f(Z_{s-} + h(\Delta X_s)))k(s, \Delta X_s)$ is strictly positive in the jump points.

$$\int_0^T \left(D_t^{\mathbb{R}_0,k} Z_T\right)^2 dt =$$

$$= \sum_{i=1}^{N_T^{\mathbb{R}_0}} \exp\left(2\int_{T_i^{\mathbb{R}_0}}^T f'(Z_u)du\right)\left(f(Z_{T_i^{\mathbb{R}_0}-}) - f(Z_{T_i^{\mathbb{R}_0}})\right)^2$$

$$\times (k(\Delta X_{T_i^{\mathbb{R}_0}}))^2 T_i^{\mathbb{R}_0}\left(1 - \frac{T_i^{\mathbb{R}_0}}{T}\right)$$

$$+2\sum_{1\leq i<j\leq N_T^{\mathbb{R}_0}} \exp\left(\int_{T_i^{\mathbb{R}_0}}^T f'(Z_u)du\right)\left(f(Z_{T_i^{\mathbb{R}_0}-}) - f(Z_{T_i^{\mathbb{R}_0}})\right) k(\Delta X_{T_i^{\mathbb{R}_0}})$$

$$\times \exp\left(\int_{T_j^{\mathbb{R}_0}}^T f'(Z_u)du\right)\left(f(Z_{T_j^{\mathbb{R}_0}-}) - f(Z_{T_j^{\mathbb{R}_0}})\right) k(\Delta X_{T_j^{\mathbb{R}_0}}) T_i^{\mathbb{R}_0}\left(1 - \frac{T_j^{\mathbb{R}_0}}{T}\right)$$

which is bigger that zero on the set $\left[N_T^{\mathbb{R}_0} \geq 1\right]$. Therefore Z_T absolutely continuous in this set.

Finally we can proceed similarly in the case that f is decreasing using the function $(h \wedge 1) \vee (-1)$ instead of k. $\square$

Case with infinite Lévy measure

Now we deal with the case that f is only monotone on an neighborhood of the initial condition x. This problem has been analyzed by Nourdin and Simon using an stratification method.

Remember the equation that we are considering:

$$Z_t = x + \int_0^t f(Z_s)ds + \int_0^t \int_{\mathbb{R}_0} h(y)dN(s,y), \quad t \in [0,T].$$

Here $x \in \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$ is a function with a bounded derivative and $h \in L^2(\mathbb{R}_0, \nu) \cap L^1(\mathbb{R}_0, \nu)$.

Theorem

Assume that $\nu(\mathbb{R}_0) = \infty$, $h(y) \neq 0$ for $y \in \mathbb{R}_0$ and that f is a monotone function on a neighborhood of the point x. Then, the random variable Z_T is absolutely continuous.

Proof: We assume that there is $\varepsilon > 0$ such that f is increasing on $(x-\varepsilon, x+\varepsilon)$ because the proof for f decreasing is similar to this one. The fact that f has a bounded derivative and the Grondwall lemma imply that there exists $M > 0$ and $t_0 \in (0, T)$ such that

$$|Z_t - x| \leq \frac{\varepsilon}{2} + e^{MT} \int_0^t \int_{\mathbb{R}_0} |h(y)| dN(s, y), \quad t \in [0, t_0].$$

Now, for $t \in [0, t_0]$, let

$$A_t = \left\{ e^{MT} \int_0^t \int_{\mathbb{R}_0} |h(y)| dN(s, y) > \frac{\varepsilon}{2} \right\}.$$

Let $k_t : [0, T] \to \mathbb{R}$ be a function in $C^1((0, T))$ such that $k_t(s) > 0$ for $s \in [0, t)$, and $k_t(s) = 0$ for $s \geq t$.

Define $((h(y) \wedge 1) \vee (-1))$. It is is bounded and the image has the same sign than $h(y)$.

Define $k^{(t)}(s, y) = -k_t(s)((h(y) \wedge 1) \vee (-1))$. If $s < t$, this function has opposite sign than $h(y)$.

As $|Z_s - x| \leq \varepsilon$ for $s \leq t \leq t_0$ on the set A_t^c, then, in this set, $(f(Z_{s-}) - f(Z_s))k^{(t)}(s, y) > 0$ if Z has a jump at time $s \in (0, t)$ (f is increasing). But, as the Lévy measure is infinite, we have, with probability 1, jumps in any neighborhood of the origin. So

$$\int_0^T \left(D_u^{\mathbb{R}_0, k^{(t)}} Z_T \right)^2 du > 0. \quad \text{on} \quad A_t^c,$$

As a consequence, by the criterium of absolute continuity, the condition $\lambda(B) = 0$ implies that

$$P([Z_T \in B] \cap A_t^c) = 0.$$

Finally, we observe that $A_t^c \subset A_{t'}^c$ for $t' < t$ and that $P(\cup_{t<t_0} A_t^c) = 1$, which follows from the Markov inequality

$$P(A_t) \leq \frac{4e^{2MT}}{\varepsilon^2} E\left(\left(\int_0^t \int_{\mathbb{R}_0} |h(y)| dN(s,y)\right)^2\right)$$

$$\leq \frac{8e^{2MT}}{\varepsilon^2} t \left(\int_{\mathbb{R}_0} |h(y)|^2 \nu(dy) + T\left(\int_{\mathbb{R}_0} |h(y)|\nu(dy)\right)^2\right).$$

Now we can conclude using that $P([Z_T \in B]) = \lim_{t\downarrow 0} P([Z_T \in B] \cap A_t^c)$, so Z_T is absolutely continuous. □

A pure discontinuous equation with no zero Wronskian

In this section we assume that the Lévy measure ν is finite and consider the stochastic differential equation

$$Z_t = x + \int_0^t f(Z_s)ds + \int_0^t \int_{\mathbb{R}_0} h(y)g(Z_{s-})dN(s,y), \quad t \in [0,T].$$

where $x \in \mathbb{R}$, $h \neq 0$ is a bounded function in $L^2(\mathbb{R}_0,\nu)$, and $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are two bounded functions with two bounded derivatives and one bounded derivative, respectively. The existence of a density for Z_T was analyzed by Carlen and Pardoux in the case that the involved Lévy process is a Poisson process. In the remaining of this section we also assume that

$$|h(y)W(g,f)(x)| > \frac{1}{2}||f''||_\infty ||h||_\infty^2 ||g||_\infty^2, \quad x \in \mathbb{R} \text{ and } y \in \mathbb{R}_0, \qquad (1)$$

where $W(g,f) = g'f - f'g$ is the Wronskian of g and f.

We consider the flow $\{\Phi_t(s,x) : 0 \leq s \leq t \leq T \text{ and } x \in \mathbb{R}\}$ associated with the equation and the family of stopping times $\{T_j : j \in \mathbb{N}\}$ of the Lévy process X. Remember that now we are dealing with a finite Lévy measure in this part of the paper.

Note that the fact that $Z_{T_n} = \Phi_{T_n}(T_{n-1}, Z_{T_{n-1}}) + h(\Delta X_{T_n}) g(Z_{T_n})$ allows to utilize induction on n to get

$$D_t^{\mathbb{R}_0,k} Z_T = \int_0^T \int_{\mathbb{R}_0} \exp\left(\int_s^T f'(Z_u)du\right) r_t(s,y) dN(s,y)$$
$$+ \sum_{m=2}^{\infty} \int_{S_m(T)} \exp\left(\int_{s_1}^T f'(Z_u)du\right) h(y_m) g'(Z_{s_m -}) \cdots h(y_2) g'(Z_{s_2 -})$$
$$\times r_t(s_1,y_1) dN(s_1,y_1) \cdots dN(s_m,y_m)$$

where

$$r_t(s,y) = k(s,y)\left(\frac{s}{T} - \mathbb{1}_{[0,s]}(t)\right)\left(f(Z_{s-}) - f(Z_s) + f(Z_{s-}) h(y) g'(Z_{s-})\right)$$

and

$$S_m(r) = \{(s_1,y_1;\ldots;s_m,y_m) \in ([0,T]\times\mathbb{R}_0)^m : 0 \leq s_1 < \cdots < s_m \leq r\}.$$

Theorem

Let Z be the solution of equation and $k \in \mathcal{K}$. Then Z_T is in the domain of $D^{\mathbb{R}_0,k}$ and has a density on the set $[N_T^{\mathbb{R}_0} > 0]$.

Proof: We first observe that $D^{\mathbb{R}_0,1}Z_T$ is a process different than 0 on $[N_T = 1]$.

Now for a rational number $p \in (0, T)$ and a positive integer $n \geq 2$, we introduce the set $A_{p,n} = [N_T = n] \cap [T_{n-1} \leq p < T_n]$ and choose a function $h_p : [0, T] \to \mathbb{R}_+$ of class C^1 such that $h_p(s) = 0$ for $s \leq p$, and $h_p(s) > 0$ for $s > p$.

Then in the set $A_{p,n}$ we only have to consider in the derivative the jump at T_n,

$$
\begin{aligned}
D_t^{\mathbb{R}_0,h_p} Z_T &= \exp\left(\int_{T_n}^T f'(Z_u)du\right) h_p(T_n)\left(\frac{T_n}{T} - \mathbb{1}_{[0,T_n]}(t)\right) \\
&\quad \times \left(f(Z_{T_n-}) - f(Z_{T_n}) + f(Z_{T_n-})h(\Delta X_{T_n})g'(Z_{T_n-})\right),
\end{aligned}
$$

which is different than zero.
Hence by the criterium of absolute continuity $P([Z_T \in B] \cap A_{p,n}) = 0$ for any Borel set $B \subset \mathbb{R}$ such that $\lambda(B) = 0$.
Thus, for this Borel set we have

$$
P([Z_T \in B] \cap [N_T > 0]) = P([Z_T \in B] \cap \cup_{n\in\mathbb{N}, p\in\mathbb{Q}} A_{p,n}) = 0,
$$

and the proof is finished.