Wavelet variations of non-linear subordinated processes with memory

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joint works with M. Clausel, F. Roueff, M. Taqqu

Let $X = \{X_n\}_{n\in\mathbb{Z}}$ be a stationary Gaussian process with mean zero, unit variance and spectral density $f(\lambda), \lambda \in (-\pi, \pi]$ and thus covariance equal to

$$r(n) = \mathbb{E}(X_0 X_n) = \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}n\lambda} f(\lambda) \mathrm{d}\lambda \,.$$

The process $\{X_n\}_{n\in\mathbb{Z}}$ is said to have :

short memory or *short–range dependence* if $f(\lambda)$ is bounded around $\lambda = 0$

long memory or *long–range dependence* if $f(\lambda) \to \infty$ as $\lambda \to 0$.

We will suppose that $\{X_n\}_{n\in\mathbb{Z}}$ has long–memory with memory parameter $d > 0$, that is,

$$f(\lambda) \sim |\lambda|^{-2d} f^*(\lambda) \text{ as } \lambda \to 0$$

where $f^*(\lambda)$ is a bounded spectral density which is continuous and positive at the origin. It is convenient to interpret this behavior as the result of a fractional integrating operation, whose transfer function reads $\lambda \mapsto (1 - \mathrm{e}^{-\mathrm{i}\lambda})^{-d}$. Hence we set

$$f(\lambda) = |1 - \mathrm{e}^{-\mathrm{i}\lambda}|^{-2d} f^*(\lambda), \quad \lambda \in (-\pi, \pi] \,. \tag{1}$$

These wavelet coefficients of X are defined by

$$W_{j,k} = \sum_{t\in\mathbb{Z}} h_j(\gamma_j k - t) X_t$$

where $\gamma_j \uparrow \infty$ as $j \uparrow \infty$ is a sequence of non–negative scale factors applied at scale j, for example $\gamma_j = 2^j$ and h_j is a **filter** whose properties will be listed below.

Many works and results on the behavior of the wavelet coefficient for a Gaussian process.

Example : fractional Brownian noise

Let B_k^H, $k \in \mathbb{Z}$ a discrete fractional Brownian motion

Its spectral density is

$$f_{B^H} = \sum_{k \in \mathbb{Z}} |\lambda + 2k\pi|^{-2H-1}$$

and then

$$f_{B^H}(\lambda) = |\lambda|^{-2d} f^*(\lambda)$$

with $d = H + \frac{1}{2}$.

We consider a non-Gaussian context :

We shall also consider a process $\{Y_t\}_{t\in\mathbb{Z}}$, not necessarily stationary but its difference $\Delta^K Y$ of order $K \geq 0$ is stationary.

Moreover, instead of supposing that $\Delta^K Y$ is Gaussian, we suppose that it is obtained as the output of a non–linear filter G with Gaussian input

$$\left(\Delta^K Y\right)_t = G(X_t), \quad t \in \mathbb{Z}\,, \tag{2}$$

where $(\Delta Y)_t = Y_t - Y_{t-1}$

and where G is a function such that $\mathbb{E}(G(X_t)) = 0$ and $\mathbb{E}(G(X_t)^2) < \infty$.
-so G can be expanded into Hermite polynomials

The sequence $\{Y_t\}_{t\in\mathbb{Z}}$ can be formally expressed as

$$Y_t = \Delta^{-K} G(X_t), \quad t \in \mathbb{Z} .$$

We will focus on the wavelet coefficients of $Y = \{Y_t\}_{t\in\mathbb{Z}}$. Since $\{Y_t\}_{t\in\mathbb{Z}}$ is random so will be its wavelet coefficients which we denote by $\{W_{j,k},\, j \geq 0,\, k \in \mathbb{Z}\}$, where j indicates the scale and k the location. These wavelet coefficients are defined by

$$W_{j,k} = \sum_{t\in\mathbb{Z}} h_j(\gamma_j k - t) Y_t$$

where $\gamma_j \uparrow \infty$ as $j \uparrow \infty$ is a sequence of non–negative scale factors applied at scale j, for example $\gamma_j = 2^j$ and h_j is a filter

Assumptions on the filter

h_j has null moments up to order $M-1$, that is, for any $m \in \{0, \cdots, M-1\}$,

$$\sum_{\ell \in \mathbb{Z}} h_j(\ell)\ell^m = 0 \,. \tag{3}$$

Therefore, since $M \geq K$, $\widehat{h}_j$ can be expressed as

$$\widehat{h}_j(\lambda) = (1-\mathrm{e}^{-\mathrm{i}\lambda})^K \widehat{h}_j^{(K)}(\lambda),$$

where $\widehat{h}_j^{(K)}$ is also a trigonometric polynomial of the form

$$\widehat{h}_j^{(K)}(\lambda) = \sum_{\tau \in \mathbb{Z}} h_j^{(K)}(\tau)\mathrm{e}^{-\mathrm{i}\lambda\tau},$$

since $h_j^{(K)}$ has finite support for any j.

Using the fact that h_j has K vanishing moments we obtain another way of expressing $W_{j,k}$, namely,

$$W_{j,k} = \sum_{\ell \in \mathbb{Z}} h_j^{(K)}(\gamma_j k - \ell) G(X_\ell) \ . \tag{4}$$

we describe the limit in distribution of the wavelet coefficients $\{W_{j+m,k}\}_{m,k}$ as $j \to \infty$, adequately normalized

Recall that $W_{j+m,k}$ involves a sum of chaoses of all order. In the limit, however, only the order q_0 (the Hermite rank) will prevail.

Spectral multiple integrals

It is convenient to use an integral representation in the spectral domain to represent the random processes.

The stationary Gaussian process $\{X_k, k \in \mathbb{Z}\}$ with spectral density (1) can be written as

$$X_\ell = \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}\lambda\ell} f^{1/2}(\lambda) \mathrm{d}\widehat{W}(\lambda) = \int_{-\pi}^{\pi} \frac{\mathrm{e}^{\mathrm{i}\lambda\ell} f^{*1/2}(\lambda)}{|1-\mathrm{e}^{-\mathrm{i}\lambda}|^{d}} \mathrm{d}\widehat{W}(\lambda), \quad \ell \in \mathbb{N} .$$

This is a special case of

$$\widehat{I}(g) = \int_{\mathbb{R}} g(x) \mathrm{d}\widehat{W}(x),$$

where $\widehat{W}(\cdot)$ is a complex–valued Gaussian random measure satisfying

$$\begin{aligned}
\mathbb{E}(\widehat{W}(A)) &= 0 \quad \text{for every Borel set } A \text{ in } \mathbb{R}\ , \\
\mathbb{E}(\widehat{W}(A)\overline{\widehat{W}(B)}) &= |A \cap B|\ , \\
\sum_{j=1}^{n} \widehat{W}(A_j) &= \widehat{W}(\bigcup_{j=1}^{n} A_j) \text{ if } A_1, \cdots, A_n \text{ are disjoint Borel sets}\ , \\
\widehat{W}(A) &= \overline{\widehat{W}(-A)}\ .
\end{aligned}$$

The integral $\hat{I}(g)$ is defined for any function $g \in L^2(\mathbb{R})$ and one has the isometry

$$\mathbb{E}(|\widehat{I}(g)|^2) = \int_{\mathbb{R}} |g(x)|^2 \mathrm{d}x \,.$$

The integral $\widehat{I}(g)$, moreover, is real–valued if

$$g(x) = \overline{g(-x)} \,.$$

We shall also consider multiple Itô–Wiener integrals

$$\widehat{I}_q(g) = \int_{\mathbb{R}^q}'' g(\lambda_1, \cdots, \lambda_q) \mathrm{d}\widehat{W}(\lambda_1) \cdots \mathrm{d}\widehat{W}(\lambda_q)$$

where the double prime indicates that one does not integrate on hyperdiagonals $\lambda_i = \pm\lambda_j, i \neq j$.

The integrals $\widehat{I}_q(g)$ are handy because we will be able to expand our non–linear functions $G(X_k)$ i in multiple integrals of this type.

These multiples integrals are as follows. Denote by $\overline{L^2}(\mathbb{R}^q, \mathbb{C})$ the space of complex valued functions defined on $\mathbb{R}^q$ satisfying

$$f(-x_1, \cdots, -x_q) = \overline{f(x_1, \cdots, x_q)} \text{ for } (x_1, \cdots, x_q) \in \mathbb{R}^q \ ,$$
$$\|f\|_{L^2}^2 := \int_{\mathbb{R}^q} |f(x_1, \cdots, x_q)|^2 \, \mathrm{d}x_1 \cdots \mathrm{d}x_q < \infty \ .$$

Let $\tilde{L}^2(\mathbb{R}^q, \mathbb{C})$ denote the set of functions in $\overline{L^2}(\mathbb{R}^q, \mathbb{C})$ that are symmetric in the sense that $f = \tilde{f}$ where $\tilde{f}(x_1, \cdots, x_q) = 1/q! \sum_\sigma f(x_{\sigma(1)}, \cdots, x_{\sigma(q)})$, where the sum is over all permutations of $\{1, \ldots, q\}$.

One defines now the multiple integral with respect to the spectral measure $\widehat{W}$ by a density argument. For a step function of the form

$$f = \sum_{j_\ell = \pm 1, \cdots, \pm N} c_{j_1, \cdots, j_n} 1_{\Delta_{j_1}} \times \cdots \times 1_{\Delta_{j_n}}$$

where the c's are real–valued, $\Delta_{j_\ell} = -\Delta_{-j_\ell}$ and $\Delta_{j_\ell} \cap \Delta_{j_m} = \emptyset$ if $\ell \neq m$, one sets

$$\widehat{I}_q(f) = \sum_{j_\ell = \pm 1, \cdots, \pm N}{}'' c_{j_1, \cdots, j_n} \widehat{W}(\Delta_{j_1}) \cdots \widehat{W}(\Delta_{j_n}) \,. \tag{5}$$

Here, $\sum''$ indicates that one does not sum over the hyperdiagonals, that is, when $j_\ell = \pm j_m$ for $\ell \neq m$.

The integral $\widehat{I}_q(f)$ verifies that

$$\mathbb{E}(\widehat{I}_q(f)\widehat{I}_{q'}(g)) = \begin{cases} q!\langle f, g\rangle_{L^2}, & \text{if } q = q' \\ 0, & \text{if } q \neq q'. \end{cases} \tag{6}$$

Observe, moreover, that for every step function f with q variables as above

$$\widehat{I}_q(f) = \widehat{I}_q(\tilde{f}).$$

Since the set of step functions is dense in $\overline{L^2}(\mathbb{R}^q, \mathbb{C})$, one can extend $\widehat{I}_q$ to an isometry from $\overline{L^2}(\mathbb{R}^q, \mathbb{C})$ to $L^2(\Omega)$ and the above properties hold true for this extension.

Remark. $\widehat{I}_q(f)$ is a real–valued random variable.

Our results are based on the expansion of the function G in Hermite polynomials. The Hermite polynomials are

$$H_q(x) = (-1)^q e^{\frac{x^2}{2}} \frac{d^q}{dx^q}\left(e^{-\frac{x^2}{2}}\right) ,$$

in particular, $H_0(x) = 1, H_1(x) = x, H_2(x) = x^2 - 1$. If X is a normal random variable with mean 0 and variance 1, then

$$\mathbb{E}(H_q(X)H_{q'}(X)) = \int_{\mathbb{R}} H_q(x)H_{q'}(x)\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-x^2/2}\mathrm{d}x = q!\delta_{q,q'} .$$

Moreover,

$$G(X)=\sum_{q=1}^{+\infty}\frac{c_q}{q!}H_q(X)\ ,$$

where the convergence is in $L^2(\Omega)$ and where

$$c_q=\mathbb{E}(G(X)H_q(X))\ .$$

q_0 =the Hermite rank of G is c_{q_0}= the first non-zero coefficients

Hermite polynomials are related to multiple integrals as follows : if $X = \int_{\mathbb{R}} g(x) \mathrm{d}\widehat{W}(x)$ with $\mathbb{E}(X^2) = \int_{\mathbb{R}} |g(x)|^2 \, \mathrm{d}x = 1$ and $g(x) = \overline{g(-x)}$ so that X has unit variance and is real–valued, then

$$H_q(X) = \widehat{I}_q(g^{\otimes q}) = \int_{\mathbb{R}^q}'' g(x_1) \cdots g(x_q) \mathrm{d}\widehat{W}(x_1) \cdots \mathrm{d}\widehat{W}(x_q) \ .$$

The expansion of G induces a corresponding expansion of the wavelet coefficients $W_{j,k}$, namely,

$$W_{j,k} = \sum_{q=1}^{+\infty} \frac{c_q}{q!} W_{j,k}^{(q)} ,$$

where by (37) one has

$$W_{j,k}^{(q)} = \sum_{\ell \in \mathbb{Z}} h_j^{(K)}(\gamma_j k - \ell) H_q(X_\ell) .$$

this an element of the qth Wiener chaos

First question :

The Gaussian sequence $\{X_n\}_{n\in\mathbb{Z}}$ is long–range dependent because its spectrum at low frequencies behaves like $|\lambda|^{-2d}$ with $d > 0$ and hence explodes at $\lambda = 0$.

What about the processes $\{H_q(X_\ell)\}_\ell$ for $q \geq 2$? What is the behavior of the spectrum at low frequencies ? Does it explodes at $\lambda = 0$?

The answer depends on the respective values of q and d.

Let us define

$$q_c = \max\{q \in \mathbb{N} \ : \ q < 1/(1-2d)\} ,$$

and

$$d(q) = qd + (1-q)/2 .$$

One has

$$d(q) > 0 \quad \text{if } q \le q_c, \quad \text{that is if } q < 1/(1-2d) .$$

The following result shows that the spectral density of $\{H_q(X_\ell)\}_{\ell\in\mathbb{Z}}$ has a different behavior at zero frequency depending on whether $q \leq q_c$ or $q > q_c$. It is long–range dependent when $q \leq q_c$ and short–range dependent when $q > q_c$.

Lemma

Let q be a positive integer. The spectral density of $\{H_q(X_\ell)\}_{\ell\in\mathbb{Z}}$ *is*

$$q!f^{(\star q)} = q!(f \star \cdots \star f)\ ,$$

where the spectral density f of $\{X_\ell\}_{\ell\in\mathbb{Z}}$ *is given in (1). Moreover the following holds :*

- (i) *If* $q \leq q_c$*, then* $\lambda^{2d(q)} f^{(\star q)}(\lambda)$ *is bounded on* $\lambda \in (0, \pi)$ *and converges to a positive number as* $\lambda \downarrow 0$.
- (ii) *If* $q > q_c$*, then* $f^{(\star q)}(\lambda)$ *is bounded on* $\lambda \in (0, \pi)$ *and converges to a positive number as* $\lambda \downarrow 0$.

Hence if $q \leq q_c$, $\{H_q(X_\ell)\}_\ell$ *has long memory with parameter* $d(q) > 0$ *whereas if* $q > q_c$, $\{H_q(X)\}_\ell$ *has a short–memory behavior.*

As $j \to \infty$, we have

$$\left\{\gamma_j^{-(d(q_0)+K)} W_{j+m,k},\ m,k \in \mathbb{Z}\right\} \stackrel{\text{fidi}}{\to} c_{q_0} \left(f^*(0)\right)^{q_0/2} \left\{Y_{m,k}^{(q_0,K)},\ m,k \in \mathbb{Z}\right\}$$

where for every positive integer q,

$$Y_{m,k}^{(q,K)} = \bar{\gamma}_m^{1/2} \int_{\mathbb{R}^q}'' \frac{\mathrm{e}^{\mathrm{i}k\bar{\gamma}_m(\zeta_1+\cdots+\zeta_q)}}{(\mathrm{i}(\zeta_1+\cdots+\zeta_q))^K} \frac{\widehat{h}_\infty(\bar{\gamma}_m(\zeta_1+\cdots+\zeta_q))}{|\zeta_1|^d \cdots |\zeta_q|^d} \mathrm{d}\widehat{W}(\zeta_1)\cdots\mathrm{d}\widehat{W}(\zeta_q)\,.$$

This is called Hermite process
if $q_0 = 1$ the limit is Gaussian (FBm) if $q_0 = 2$ the limit is "Rosenblatt" (non-Gaussian)

We will separate the Wiener chaos expansion (23) of $W_{j,k}$ into two terms depending on the position of q with respect to q_c. The first term includes only the q's for which $H_q(x)$ exhibits long–range dependence (LD), that is,

$$W_{j,k}^{(LD)} = \sum_{q=0}^{q_c} \frac{c_q}{q!} W_{j,k}^{(q)} , \qquad (7)$$

and the second term includes the terms which exhibit short–range dependence (SD)

$$W_{j,k}^{(SD)} = \sum_{q=q_c+1}^{\infty} \frac{c_q}{q!} W_{j,k}^{(q)} . \qquad (8)$$

The two following results provide the asymptotic behavior of each term of the sum

The first result concerns the terms with long memory, that is, with $q \leq q_c$. The second result concerns the terms with short memory for which $q > q_c$.

Proposition

Suppose that $q \in \{1, \dots, q_c\}$. Then, as $j \to \infty$,

$$\left(\gamma_j^{-(d(q)+K)} W_{j+m,k}^{(q)},\ m,k \in \mathbb{Z}\right) \xrightarrow{fidi} \left((f^*(0))^{q/2}\ Y_{m,k}^{(q,K)},\ m,k \in \mathbb{Z}\right), \tag{9}$$

where $Y_{m,k}^{(q,K)}$ is the Hermite process.

Proposition

We have, for any $k \in \mathbb{Z}$, as $j \to \infty$,

$$W^{(SD)}_{j+m,k} = O_P(\gamma_j^K) \ . \tag{10}$$

It follows from this Proposition that the dominating term is given by the chaos of order $q = q_0$. Now, since $d(q_0) > 0$ by (25), we get that, for all (k, m), as $j \to \infty$,

$$W^{(SD)}_{j+m,k} = o_p(\gamma_j^{d(q_0)+K}) \ .$$

Proof of the convergence of the long memory term

Let $q \in \mathbb{N}^*$. For any j

$$W_{j+m,k}^{(q)} \overset{(\text{fidi})}{=} \sum_{s=-[q/2]}^{[q/2]} W_{m,k}^{(j,q,s)} , \tag{11}$$

where $[a]$ denotes the integer part of a, and for any $q \in \mathbb{N}^*$, $s \in \mathbb{Z}$,

$$W_{m,k}^{(j,q,s)} = \int_{\zeta \in \mathbb{R}^q}^{''} 1_{\Gamma(q,s)}(\gamma_j^{-1}\zeta) f_{m,k}(\zeta; j, q) \, \mathrm{d}\widehat{W}(\zeta_1) \cdots \mathrm{d}\widehat{W}(\zeta_q) , \tag{12}$$

where $f_{m,k}(\zeta; j, q)$ is defined by (setting $\xi = \gamma_j^{-1}\zeta$)

$$f_{m,k}(\gamma_j \xi; j, q) = \gamma_j^{-q/2} \frac{\exp \circ \Sigma_q(\mathrm{i}\gamma_{j+m} k\xi) \times \widehat{h}_{j+m} \circ \Sigma_q(\xi)}{\{1 - \exp \circ \Sigma_q(-\mathrm{i}\xi)\}^K} \left(f^{\otimes q}(\xi)\right)^{1/2} .$$

and where

$$\Gamma^{(q,s)} = \left\{ \xi \in (-\pi, \pi]^q, \ -\pi + 2s\pi < \sum_{i=1}^{q} \xi_i \leq \pi + 2s\pi \right\} .$$

the following convergence results, valid for all fixed $m, k \in \mathbb{Z}$ as $j \to \infty$. For $s = 0$,

$$\gamma_j^{-(d(q)+K)} W_{m,k}^{(j,q,0)} \xrightarrow{L^2} (f^*(0))^{q/2} \; Y_{m,k}^{(q,K)} \; ,$$

whereas for other values of s, namely for all $s \in \{-[q/2], \ldots, -1, 1, \ldots, [q/2]\}$,

$$\gamma_j^{-(d(q)+K)} W_{m,k}^{(j,q,s)} \xrightarrow{L^2} 0 \; ,$$

where $d(q)$ is defined in (25).

The scalogram

Our goal is to find the distribution of the empirical quadratic mean of these wavelet coefficients at large scales $j \to \infty$, that is, the asymptotic behavior of the scalogram

$$S_{n,j} = \frac{1}{n} \sum_{k=0}^{n-1} W_{j,k}^2 \ , \tag{13}$$

adequately normalized as the sample size n and $j = j(n) \to \infty$. This is a necessary and important step in developing methods for estimating the underlying long memory parameter d.

A first intuition : it comes from the study of wavelet coefficient of the Rosenblatt process

$$R_t = I_2(L_t)$$

in this case $S_{n,j} = \frac{1}{n}\sum_{k=1}^{n} W_{j,k}^2 = I_4(g_{n,j}^1) + I_2(g_{n,j}^2$

The dominant term is I_2 and it converges to a Rosenblatt distribution

Recall

$$W_{j,k} = \sum_{t\in\mathbb{Z}} h_j^{(K)}(\gamma_j k - t) G(X_t) \ ,$$

or

$$W_{j,k} = \sum_{q=1}^{+\infty} \frac{c_q}{q!} W_{j,k}^{(q)} \ ,$$

where by (37) one has

$$W_{j,k}^{(q)} = \sum_{t\in\mathbb{Z}} h_j^{(K)}(\gamma_j k - t) H_q(X_t).$$

this is a multiple integral of order q

for any $(n,j) \in \mathbb{N}^2$,

$$S_{n,j} - \mathbb{E}(S_{n,j}) = \frac{1}{n}\sum_{k=0}^{n-1} W_{j,k}^2 - \mathbb{E}[W_{j,0}^2]$$

$$= \sum_{q=1}^{\infty} \left(\frac{c_q}{q!}\right)^2 \sum_{p=0}^{q-1} p! \binom{q}{p}^2 (2\pi)^p S_{n,j}^{(q,q,p)}$$

$$+ 2\sum_{q=2}^{\infty} \sum_{q'=1}^{q-1} \frac{c_q}{q!}\frac{c_{q'}}{q'!} \sum_{p=0}^{q'} (2\pi)^p \, p! \binom{q}{p}\binom{q'}{p} S_{n,j}^{(q,q',p)} ,$$

where, for all $q, q' \geq 1$ such that $q + q' - 2p \geq 1$,

$$S_{n,j}^{(q,q',p)} = \widehat{I}_{q+q'-2p}(g) \ .$$

We call $q + q' - 2p$ the *order* of the summand $S_{n,j}^{(q,q',p)}$. The function $g(\xi)$, $\xi = (\xi_1, \dots, \xi_{q+q'-2p}) \in \mathbb{R}^{q+q'-2p}$ is defined for every p, q, q' with $q + q' - 2p \geq 1$ as

$$g(\xi) = \frac{1}{n} \sum_{k=0}^{n-1} \left(f_{j,k}^{(q)} \overline{\otimes}_p f_{j,k}^{(q')} \right) \ ,$$

where the operation $\overline{\otimes}_p$ is a contraction from product formula for multiple integrals

We have an infinite number of summand

in each chaos there an infinite number of summands

We set

$$I = \{\ell \in \mathcal{L} \; : \ell + 1 \in \mathcal{L}, q_{\ell+1} - q_\ell = 1\} \; ,$$
$$J = \{(\ell_1, \ell_2) \in \mathcal{L}^2 \; : \; \ell_1 < \ell_2 \text{ and } q_{\ell_2} - q_{\ell_1} \geq 2\} \; ,$$

that is, q_ℓ and $q_{\ell+1}$ takes consecutive values when $\ell \in I$ and q_{ℓ_1} and q_{ℓ_2} differ by two or more when $(\ell_1, \ell_2) \in J$. The structure of I is particularly important. The set I could be empty (there are no consecutive values of q_ℓ) or not empty. If it is not empty, then we set

$$\ell_0 = \min(I) \geq 0$$

and q_{ℓ_0}, the smallest index q such that two Hermite coefficients c_q, c_{q+1} are non–zero, will be involved in the normalization.

For example,

$$\text{if } c_1 \neq 0,\ c_2 \neq 0,\ c_3 = 0, c_4 \neq 0, c_q = 0 \text{ for } q \geq 5\ ,$$

then $I = \{1\}$, $\ell_0 = 1$, $J = \{(1,4),(2,4)\}$ whereas

$$\text{if } c_1 = 0,\ c_2 \neq 0,\ c_3 \neq 0, c_4 \neq 0, c_q = 0 \text{ for } q \geq 5\ ,$$

then $I = \{2,3\}$, $\ell_0 = 2$ and $J = \{(2,4)\}$. If $c_1 \neq 0$ and $c_q = 0$ for $q \geq 2$, then both I and J are empty.

The asymptotic behavior of $S_{n,j}$, centered by its mean $\mathbb{E}(|W_{j+m,0}|^2)$, depends on the sets J and I.

This concerns both the rate of convergence and the limit distribution of the rescaled sequence.

The expansion in Wiener chaos of $S_{n,j}$ implies that

$$S_{n,j} - \mathbb{E}(|W_{j,0}|^2) = T_{n,j}^{(0)} + T_{n,j}^{(1)} + T_{n,j}^{(2)} , \tag{14}$$

with

$$T_{n,j}^{(0)} = \sum_{\ell \in \mathcal{L}} \frac{c_{q_\ell}^2}{(q_\ell !)^2} \sum_{p=0}^{q_\ell - 1} p! \binom{q_\ell}{p}^2 S_{n,j}^{(q_\ell, q_\ell, p)} ,$$

$$T_{n,j}^{(1)} = \sum_{(\ell_1,\ell_2) \in J} \frac{c_{q_{\ell_1}}}{q_{\ell_1}!} \frac{c_{q_{\ell_2}}}{q_{\ell_2}!} \sum_{p=0}^{q_{\ell_1}} p! \binom{q_{\ell_1}}{p} \binom{q_{\ell_2}}{p} S_{n,j}^{(q_{\ell_1}, q_{\ell_2}, p)} ,$$

and

$$T_{n,j}^{(2)} = 2\sum_{\ell\in I} \frac{c_{q_\ell+1}}{(q_\ell+1)!}\frac{c_{q_\ell}}{q_\ell!}\sum_{p=0}^{q_\ell} p!\binom{q_\ell+1}{p}\binom{q_\ell}{p} S_{n,j}^{(q_\ell+1,q_\ell,p)} .$$

Let us comment on this decomposition . The sum $T_{n,j}^{(0)}$ contains squares of multiple integrals of orders q_ℓ.

The multiplication formula for multiple integrals implies that this sum, after subtracting its expectation, has only summands of order $2, 4, 6, \ldots$ in the Wiener chaos.

The sum $T_{n,j}^{(1)}$ contains product of multiple integrals of orders q_{ℓ_1} and q_{ℓ_2} with $(\ell_1, \ell_2) \in J$. **That means that all the summands in $T_{n,j}^{(1)}$ are of order greater or equal than 2.**

The last sum $T_{n,j}^{(2)}$ contains **possibly components in the first Wiener chaos, hence Gaussian terms.** It is therefore natural to study $T_{n,j}^{(0)} + T_{n,j}^{(1)}$ and $T_{n,j}^{(2)}$ separately.

The general phenomenon :
in $T^{(2)}_{n,j}$ the dominant term is the chaos of order 1 (Gaussian) ; but there are many terms in the chaos of order 1

For example

$$I_2(f)I_3(g) = I_5(f \otimes g) + I_3(f \otimes_1 g) + I_1(f \otimes_2 g)$$

and

$$I_3(f') \times I_4(g') = I_7(f' \otimes g') + I_5(f' \otimes_1 g') + I_3(f' \otimes_2 g') + I_1(f' \otimes_3 g')$$

Between $I_1(f \otimes_2 g)$ and $I_1(f' \otimes_3 g')$ the first is the dominant term

We will see that $T_{n,j}^{(0)} + T_{n,j}^{(1)}$ will converge to a non-Gaussian limit (actually a random variable in the second Wiener chaos), whereas $T_{n,j}^{(2)}$ will tend to a Gaussian limit.

We will estimate separately the asymptotic behavior of $T_{n,j}^{(0)} + T_{n,j}^{(1)} - \mathbb{E}(|W_{j,0}^2|)$ and $T_{n,j}^{(2)}$.

We shall now focus on the potential leading terms.

Proposition

1. *Assume that* $d > 1/4$. *Then when* $j, n \to \infty$

$$\left(n^{1-2d}\gamma_j^{-2(\delta(q_0)+K)}(T^{(0)}_{n,j+m} + T^{(1)}_{n,j+m}), m \in \mathbb{Z}\right) \overset{fidi}{\to} (R_m, m \in \mathbb{Z}).$$

   (R_m is a Rosenblatt r.v, in the second Wiener chaos)

2. *Assume that* $d \leq 1/4$. *Then*

$$\sup_{n,j}\left(n^{1/2}\gamma_j^{-2(\delta(q_0)+K)}\left(\|T^{(0)}_{n,j}\|_2 + \|T^{(1)}_{n,j}\|_2\right)\right) < \infty\,.$$

We now want to identify the leading term for $T_{n,j}^{(2)}$.

Proposition

Under certain hypothesis, if $q_{\ell_0} + 1 < 1/(1-2d)$ *then*

$$\left(n^{(1-2d)/2} \gamma_j^{-(\delta(q_{\ell_0})+\delta(q_{\ell_0}+1)+2K)} T_{n,j+m}^{(2)}, m \in \mathbb{Z} \right) \overset{fidi}{\longrightarrow} (G_m, m \in \mathbb{Z})$$

G_m is Gaussian (fractional Brownian motion)

Let

$$C_0(j,n) = \frac{n^{1-2d}\gamma_j^{-(2\delta(q_0)+2K)}}{n^{(1-2d)/2}\gamma_j^{-(\delta(q_{\ell_0})+\delta(q_{\ell_0}+1)+2K)}} = \frac{n^{(1-2d)/2}}{\gamma_j^{(2q_{\ell_0}+1-2q_0)(1-2d)/2}}. \tag{15}$$

we have three mutually exclusive cases which cover all possibilities provided that the limit of $C_0(j,n)$ when $j,n \to +\infty$ exists (it may be infinite).

-there is game between n and γ_j

Recall that $K \geq 0$ is the number of vanishing moments, q_0 is the Hermite rank , $\delta(q)$, ℓ_0 is min I
Three limits are possible. They involve :
R_m =the Rosenblatt process and
G_m = the Gaussian fractional Brownian motion.

Then

a. Assume that $q_0 \geq 2$. If either I is empty or if

$$I \text{ is not empty }, q_{\ell_0+1} < 1/(1-2d) \text{ and } \lim_{j,n\to\infty} C_0(j,n) = 0 \,,$$

then, as $j, n \to \infty$,

$$\left(n^{1-2d} \gamma_j^{-2(\delta(q_0)+K)} \left(S_{n,j+m} - \mathbb{E}(|W_{j+m,0}|^2) \right), m \in \mathbb{Z} \right) \stackrel{\text{fidi}}{\longrightarrow} (R_m, m \in \mathbb{Z}) \,.$$

b. If I is not empty, $q_{\ell_0+1} < 1/(1-2d)$ and

$$\lim_{j,n\to\infty} C_0(j,n) = \infty \ ,$$

then, as $j, n \to \infty$,

$$\left(n^{(1-2d)/2}\gamma_j^{-(\delta(q_{\ell_0})+\delta(q_{\ell_0}+1)+2K)}\left(S_{n,j+m}-\mathbb{E}(|W_{j+m,0}|^2)\right), m\in\mathbb{Z}\right)$$
$$\xrightarrow{\text{fidi}} (G_m, m\in\mathbb{Z}) \ .$$

c. Assume that $q_0 \geq 2$. If I is not empty, $q_{\ell_0+1} < 1/(1-2d)$ and

$$\lim_{j,n\to\infty} C_0(j,n) = C_0 \ ,$$

with $C_0 > 0$, then, as $j, n \to \infty$,

$$\left(n^{1-2d}\gamma_j^{-2(\delta(q_0)+K)} \left(S_{n,j+m} - \mathbb{E}(|W_{j+m,0}|^2) \right), m \in \mathbb{Z} \right) \xrightarrow{\text{fidi}} (Y_m, m \in \mathbb{Z}) \ ,$$

where for any $m \in \mathbb{Z}$,

$$\begin{aligned} Y_m &= R_m + C_0 G_m \\ &= C_1 \int''_{\mathbb{R}^2} \frac{\mathrm{e}^{\mathrm{i}(u_1+u_2)\bar{\gamma}_m} - 1}{\mathrm{i}(u_1+u_2)} |u_1 u_2|^{-d} \mathrm{d}\widehat{W}(u_1)\mathrm{d}\widehat{W}(u_2) \\ &\quad + C_2 \int''_{\mathbb{R}} \frac{\mathrm{e}^{\mathrm{i}u\bar{\gamma}_m} - 1}{\mathrm{i}u} |u|^{-d} \mathrm{d}\widehat{W}(u) \ . \end{aligned} \tag{16}$$

Remark 3.1 Observe that $C_0(j, n)$ involves a comparison of the normalization factors in cases (a.),(b.),(c.) when I is not empty. If I is empty, there is no leading Gaussian terms which may compete with the other terms, and hence no need to compare normalizations. Note that there may still be non–leading asymptotically Gaussian terms in this case. These extra–Gaussian terms turn out to be negligible in the limit.

We will now illustrate the main result through four examples :
(i) $G = H_{q_0}$, $q_0 \geq 2$,
(ii) $G = H_{q_0} + H_{q_1}$, $q_0 \geq 2$ with $q_1 - q_0 \geq 2$,
(iii) $G = H_{q_0} + H_{q_0+1}$, $q_0 \geq 2$, $q_0 + 1 < 1/(1-2d)$ and
(iv) $G = H_{q_0} + H_{q_0+1} + H_{q_1}$, $q_0 \geq 2$, $q_0 + 1 < 1/(1-2d)$ with $q_1 - (q_0+1) \geq 2$ where in both cases q_0 is an integer. We assume that $d \in (1/4, 1/2)$.