The Malliavin chaotic derivative for Lévy processes and some recent applications

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ABSTRACT I

Itô (1956) showed that Lévy processes enjoy the chaotic representation property in a generalized form.

In other words, the space of square integrable functionals adapted to the filtration of an independent random measure associated to a Lévy process has Fock space structure.

This allows to develop a formal calculus where gradient and divergence dual operators are the main tools.

In this talk I am going to present a probabilistic interpretation of these operators based on **Solé, Utzet and Vives (2007)**. This interpretation generalizes the interpretation given in Nualart and Vives (1990) for the standard Poisson case.

ABSTRACT II

As an application I am going to comment two recent consequences of this probabilistic interpretation.

The first one, based on **Alòs, León and Vives (2008)** is an anticipating Itô formula that extends both the usual adapted formula for Lévy processes and the anticipative version of the Itô formula on the Wiener space.

The second one, based in **Alòs, León, Pontier and Vives (2008)**, is a Hull and White formula for some general stochastic volatility jump diffusion price model.

OUTLINE

1. LÉVY PROCESSES
2. FORMAL CALCULUS BASED ON THE FOCK SPACE STRUCTURE
3. A CANONICAL SPACE FOR LÉVY PROCESSES
4. PROBABILISTIC INTERPRETATION OF OPERATORS
5. FIRST APPLICATION: AN ANTICIPATING ITÔ FORMULA
6. SECOND APPLICATION: A HULL AND WHITE FORMULA FOR AN SV-JUMP-DIFFUSION PRICE MODEL

Lévy-Itô representation I

Consider a Lévy process with triplet (γ, σ, ν) where $\gamma \in \mathbb{R}$, $\sigma > 0$ and ν, the so called Lévy measure, is a Radon measure such that with $\nu(0) = 0$ and

$$\int_{\mathbb{R}} (1 \wedge x^2)\nu(dx) < \infty.$$

According to the well-known Lévy-Itô representation we have:

$$X_t = \gamma t + \sigma W_t + J_t$$

where W is the standard Brownian motion and J is a *pure jump* Lévy process, independent of W.

THE LÉVY-ITÔ REPRESENTATION II

The process J can be represented as follows:

$$J_t = \int_0^t \int_{\{|x|>1\}} x \, dN(s,x) + \lim_{\varepsilon \downarrow 0} \int_0^t \int_{\{\epsilon<|x|\leq 1\}} x \, d\widetilde{N}(s,x),$$

where $N(B) = \#\{t : (t, \Delta X_t) \in B\}$, for $B \in \mathcal{B}((0,\infty) \times \mathbb{R}_0)$, is the jump measure of the process, $d\widetilde{N}(t,x) = dN(t,x) - dt\, d\nu(x)$ is the compensated jump measure and $\mathbb{R}_0 := \mathbb{R} - \{0\}$. The limit is *a.s.* uniform in t on every bounded interval.

Moreover, if $\{\mathcal{F}_t^W, \, t \geq 0\}$ is the completed natural filtration of $\{W_t, \, t \geq 0\}$ and $\{\mathcal{F}_t^J, \, t \geq 0\}$ the completed natural filtration of J, we have, for every $t \geq 0$, $\mathcal{F}_t^X = \mathcal{F}_t^W \vee \mathcal{F}_t^J$.

The associated centered independent random measure I

From Itô (1956), X can be extended to a centered and independent random measure M on $\mathbb{R}_+ \times \mathbb{R}$.

We consider the continuous measure $\mu(dt, dx) = \eta(dx)dt$, where

$$\eta(dx) := \sigma^2 \delta_0(dx) + x^2 \, \nu(dx).$$

More explicitly, we have, for any $E \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R})$,

$$\mu(E) = \sigma^2 \int_{E(0)} dt + \iint_{E'} x^2 d\nu(x) dt,$$

where $E(0) = \{t \in \mathbb{R}_+ : (t, 0) \in E\}$ and $E' = E - \{(t, 0) \in E\}$.

THE ASSOCIATED CENTERED INDEPENDENT RANDOM MEASURE II

Then, for $E \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R})$ with $\mu(E) < \infty$, we define the measure

$$M(dt, dx) = \sigma W(dt)\delta_0(dx) + x d\widetilde{N}(t, x),$$

that is,

$$M(E) = \sigma \int_{E(0)} dW_t + \iint_{E'} x\, d\widetilde{N}(t, x),$$

that is a centered independent random measure such that

$$E\left[M(E_1)M(E_2)\right] = \mu(E_1 \cap E_2),$$

for $E_1, E_2 \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R})$ with $\mu(E_1) < \infty$ and $\mu(E_2) < \infty$.

An alternative approach I

From Løkka (2004) and Petrou (2008), for square integrable Lévy processes, a similar approach can be considered taking

$$\bar{\eta}(dx) := \sigma^2 \delta_0(dx) + \nu(dx)$$

and

$$\bar{M}(dt, dx) := \sigma W(dt)\delta_0(dx) + d\widetilde{N}(t, x).$$

MULTIPLE STOCHASTIC INTEGRALS I

Let

$$L_n^2 := L^2\Big((\mathbb{R}_+ \times \mathbb{R})^n, \mathcal{B}(\mathbb{R}_+ \times \mathbb{R})^n, \mu^{\otimes n}\Big).$$

For $f \in L_n^2$, following Itô, we can define a multiple stochastic integral $I_n(f)$ with respect M, through the same steps as in the Wiener case.

For $f = \mathbf{1}_{E_1 \times \cdots \times E_n}$, where $E_1, \ldots, E_n \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R})$, pairwise disjoints, with $\mu(E_1) < \infty, \ldots, \mu(E_n) < \infty$, we define $I_n(f) := M(E_1) \cdots M(E_n)$ and then, this definition is extended to L_n^2 by linearity and continuity.

Multiple stochastic integrals II

This integral has the usual properties:

1. $$I_n(f) = I_n(\widetilde{f}),$$
   where $\widetilde{f}$ is the symmetrization of f
2. $$I_n(af + bg) = aI_n(f) + bI_n(g).$$
3. $$E[I_n(f)I_m(g)] = \delta_{n,m} n! \int_{(\mathbb{R}_+ \times \mathbb{R})^n} \widetilde{f}\, \widetilde{g}\, d\mu^{\otimes n},$$
   where $\delta_{n,m} = 1$, if $n = m$, and 0 otherwise.

Chaotic representations property

Itô (1956) proved that

$$L^2(\Omega, \mathcal{F}^X) = \bigoplus_{n=0}^{\infty} I_n(L_n^2),$$

Then we can represent any functional $F \in L^2(\Omega, \mathcal{F}^X)$ via the expansion

$$F = \sum_{n=0}^{\infty} I_n(f_n), \quad f_n \in L_n^2.$$

This expansion is unique if we take every f_n symmetric.

This fact makes possible to apply the machinery of anhilation and creation operators in a Fock space as presented for example in Nualart and Vives (1990).

Gradient operator I

Let $F \in L^2(\Omega)$, with chaotic representation $F = \sum_{n=0}^{\infty} I_n(f_n)$, ($f_n$ symmetric) and such that $\sum_{n=1}^{\infty} n\, n! \|f_n\|^2_{L^2_n} < \infty$.

The Malliavin derivative of such a F is

$$D_z F = \sum_{n=1}^{\infty} n I_{n-1}\Big(f_n\big(z, \cdot\big)\Big), \quad z \in \mathbb{R}_+ \times \mathbb{R},$$

in $L^2(\mathbb{R}_+ \times \mathbb{R} \times \Omega, \mu \otimes \mathbb{P})$.

Denote by $\mathrm{Dom} D$ the domain of the operator D.

GRADIENT OPERATOR II

In particular we can also define

$$D_{t,0}F = \sum_{n=1}^{\infty} nI_{n-1}\Big(f_n\big((t,0),\cdot\big)\Big), \quad t \in \mathbb{R}_+,$$

in $L^2(\mathbb{R}_+ \times \Omega, dt \otimes \mathbb{P})$ and

$$D_z F = \sum_{n=1}^{\infty} nI_{n-1}\Big(f_n\big(z,\cdot\big)\Big), \quad z = (t,x) \in \mathbb{R}_+ \times \mathbb{R}_0,$$

in $L^2(\mathbb{R}_+ \times \mathbb{R}_0 \times \Omega, dt\, x^2 d\nu(x) \otimes \mathbb{P})$.

Define respectively its domains $\mathrm{Dom} D^0$ and $\mathrm{Dom} D^J$. When both $\sigma > 0$ and $\nu \neq 0$, then $\mathrm{Dom} D = \mathrm{Dom} D^0 \cap \mathrm{Dom} D^J$.

THE DIVERGENCE OPERATOR

Let $f \in L^2(\mathbb{R}_+ \times \mathbb{R} \times \Omega, \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}) \otimes \mathcal{F}^X, \mu \otimes \mathbb{P})$. We have the chaotic decomposition

$$f(z) = \sum_{n=0}^{\infty} I_n(f_n(z, \cdot))$$

where $f \in L^2_{n+1}$ is symmetric in the n last variables. Let $\widehat{f}_n$ be the symmetrization in all $n+1$ variables.
Then we define the Skorohod integral of f by

$$\delta(f) = \sum_{n=0}^{\infty} I_{n+1}(\widehat{f}_n),$$

in $L^2(\Omega)$, provided $f \in \operatorname{Dom} \delta$, that means

$$\sum_{n=0}^{\infty}(n+1)! \, \|\widehat{f}_n\|^2_{L^2_{n+1}} < \infty.$$

Duality between D and δ.

If $f \in \mathrm{Dom}\,\delta$ and $F \in \mathrm{Dom}\,D$ we have

$$E[\delta(f)\,F] = E\iint_{\mathbb{R}_+\times\mathbb{R}} f(z)\,D_zF\,\mu(dz).$$

Alternative approach II

The same development can be obtained under the Løkka - Petrou approach, using $\bar{\mu}$, defining multiple stochastic integrals $\bar{I}_n$ with respect to $\bar{M}$, and so on.

Observe that, if $h(x) = x\mathbf{1}_{\{x\neq 0\}} + \mathbf{1}_{\{x=0\}}$, we have

$$I_n(f_n) = \bar{I}_n(g_n), \text{ with } f_n = g_n h^{\otimes n},$$

$$\bar{D}_z F = h(x) D_z F, \text{ for } z = (t, x),$$

$$\bar{\delta}(u) = \delta(\frac{u}{h}),$$

and the duality between $\bar{D}$ and $\bar{\delta}$.

A new canonical space for Lévy processes

The usual canonical Lévy process is built on the space of measurable functions from $\mathbb{R}_+$ to $\mathbb{R}$ or on the space of *cadlag* functions, in both cases with the σ-field generated by the cylinders and using the Kolmogorov extension theorem.

In order to have a probabilistic interpretation of the operator *D*, in Solé, Utzet and Vives (2007) a different canonical Lévy process, that is an extension of the canonical Poisson process defined in Neveu (1977), is constructed.

STEPS OF THE CONSTRUCTION

We are going to construct a canonical space for Lévy processes by the following steps:

1. A canonical space for a compound Poisson process in a finite time interval.
2. A canonical space for a compound Poisson process in an infinite time interval.
3. A canonical space for pure jump Lévy processes.
4. A canonical space for Lévy processes.

A canonical space for a compound Poisson process in a finite time interval I

Fix $T > 0$ and consider a compound Poisson process of the form

$$Y_t = \sum_{j=1}^{N_t} Z_j, \quad t \in [0, T],$$

where $\{N_t, \, t \in [0, T]\}$ is a λ-Poisson process and $\{Z_n, \, n \geq 1\}$ is a sequence of i.i.d. random variables with law Q, supported on $S \in \mathcal{B}(\mathbb{R}_0)$.

Any trajectory of Y is totally described by a finite sequence $\big((t_1, x_1), \ldots, (t_n, x_n)\big)$ where $t_1, \ldots, t_n \in [0, T]$ are the jump instants, and $x_1, \ldots, x_n \in \mathbb{R}_0$ are its sizes.

A canonical space for a compound Poisson process in a finite time interval II

We define

- $\Omega_T = \bigcup_{n\geq 0} \left([0,T]\times S\right)^n$, where $\left([0,T]\times S\right)^0 = \{\alpha\}$, being α a distinguished element that represents the empty sequence.

- $\mathcal{F}_T = \left\{B \subset \Omega_T : \; B = \bigcup_{n\geq 0} B_n, \; B_n \in \mathcal{B}\big([0,T]\times S\big)^n\right\}$ $\bigvee_{n\geq 0} \mathcal{B}\big(([0,T]\times S)^n\big)$.
- P_T such that for $B = \bigcup_n B_n, \; B_n \in \mathcal{B}\big([0,T]\times S)^n$,

$$P_T(B) = e^{-\lambda T}\sum_{n=0}^{\infty} \frac{\lambda^n (\ell\otimes Q)^{\otimes n}(B_n)}{n!},$$

where $(\ell\otimes Q)^0 = \delta_\alpha$.

A canonical space for a compound Poisson process in a finite time interval III

Then we define $\{X_t,\ t \in [0,T]\}$ by

$$X_t(\omega) = \begin{cases} \sum_{j=1}^{n} x_j \, \mathbf{1}_{[0,t]}(t_j), & \text{if } \omega = \big((t_1,x_1),\ldots,(t_n,x_n)\big), \\ 0, & \text{if } \omega = \alpha. \end{cases}$$

For $\omega = \big((t_1,x_1),\ldots,(t_n,x_n)\big)$, the trajectory $X_\cdot(\omega)$ on $[0,T]$ has jumps at, and only at, the points $t_1,\ldots,t_n$ of sizes $x_1,\ldots,x_n$ respectively. Note that $t_1,\ldots,t_n$ not need to be increasing. When $\omega = \alpha$, then $X_\cdot(\omega)$ does not jump at all on $[0,T]$.

With the above definitions, $\{X_t,\ t \in [0,T]\}$ becomes a cadlag compound Poisson process of Lévy measure λQ.

A canonical space for a compound Poisson process in a finite time interval IV

- Given a measurable space $(E, \mathcal{E})$, it is easy to see that

$$\mathcal{E}^n_{\text{sym}} = \{C \in \mathcal{E}^{\otimes n} : \ C \text{ is symmetric}\}$$

  is a σ-field.
- A function $f : E^n \longrightarrow \mathbb{R}$ is $\mathcal{E}^n_{\text{sym}}$ measurable if and only if f is $\mathcal{E}^n$ measurable and symmetric.
- Let $\mathcal{F}_{T,\text{sym}}$ be the sub-σ-field of $\mathcal{F}_T$ defined by $\mathcal{F}_{T,\,\text{sym}} = \bigvee_{n \geq 0} \mathcal{B}([0,T] \times S)^n_{\text{sym}}$. Let $\mathcal{F}^X_T$ the σ-field generated by $\{X_t, \ t \in [0,T]\}$. We have

$$\mathcal{F}^X_T = \mathcal{F}_{T,\,\text{sym}}.$$

A canonical space for a compound Poisson process in an infinite time interval I

We extend now the construction given above to a canonical compound Poisson process, with the same λ and Q, on the whole time interval $\mathbb{R}_+$ through a projective system of probability spaces.

First of all observe that the space $\bigcup_{n\geq 0}\left([0,T]\times S\right)^n$ is a metric, separable and complete space with the distance

$$d(v,w)=\begin{cases}1, & \text{if } \ n\neq m, \ \text{ or } \ n=m \ \text{ and } \ d_{2n}(v,w)>1,\\ d_{2n}(v,w), & \text{if } \ n=m \ \text{ and } \ d_{2n}(v,w)\leq 1,\end{cases}$$

where d_k is the Euclidean distance on $\mathbb{R}^k$.
The σ-field $\bigvee_{n\geq 0}\mathcal{B}([0,T]\times S)^n$ coincides with the Borel σ-field.

A canonical space for a compound Poisson process in an infinite time interval II

For $m \geq 1$, let $(\Omega_m, \mathcal{F}_m, P_m)$ be the canonical space corresponding to the interval $[0, m]$. Consider the maps

$$\pi_m \colon \Omega_{m+1} \longrightarrow \Omega_m$$

defined by

$$\pi_m\big((t_1, x_1), \ldots, (t_r, x_r)\big) = \big((t_{i_1}, x_{i_1}), \ldots, (t_{i_s}, x_{i_s})\big),$$

where $t_{i_1}, \ldots, t_{i_s}$ are the points of $t_1, \ldots, t_r$ such that belong to $[0, m]$, and

$$\pi_m\big((t_1, x_1), \ldots, (t_r, x_r)\big) = \alpha$$

if $t_1, \ldots, t_r \in (m, m+1]$.

A canonical space for a compound Poisson process in an infinite time interval III

It is straightforward to check that

$$\mathbb{P}_m(B) = \mathbb{P}_{m+1}(\pi_m^{-1}(B)), \ \forall B \in \mathcal{F}_m.$$

From Parthasaraty (1965) there is a unique probability $\mathbb{P}$ on $(\Omega, \mathcal{F})$, where Ω is the projective limit of the system $(\Omega_m, \pi_m, m \geq 1)$, $\mathcal{F}$ is the σ-field generated by the canonical projections $\overline{\pi}_m : \Omega \to \Omega_m$, and

$$\mathbb{P}_m(B) = \mathbb{P}(\overline{\pi}_m^{-1}(B)), \ \forall B \in \mathcal{F}_m.$$

A CANONICAL SPACE FOR A COMPOUND POISSON PROCESS IN AN INFINITE TIME INTERVAL IV

By construction, the projective limit Ω is the set of all sequences $(\omega_1, \omega_2, \ldots,)$ with $\omega_m \in \Omega_m$ such that $\pi_m(\omega_{m+1}) = \omega_m$.

In our setup, Ω becomes equivalent to the set with the following elements:

- The empty sequence α, corresponding to the element $(\alpha, \alpha, \ldots)$.
- All finite sequences $((t_1, x_1), \ldots, (t_r, x_r))$, corresponding to the elements $(\omega_1, \omega_2, \ldots,)$ such that $\omega_r = \omega_{r+1}, \ldots$ for some r.
- All infinite sequences $((t_1, x_1), (t_2, x_2), \ldots)$ such that for every $T > 0$, there is only a finite number of $t_i \leq T$.

A canonical space for a compound Poisson process in an infinite time interval V

Now define the σ-field on Ω :

$$\mathcal{F}_{\text{sym}} = \bigvee_{n \geq 0} \overline{\pi}_m^{-1} \left(\mathcal{F}_{m,\text{sym}} \right).$$

Finally we have:

The process $\{X_t, \ t \geq 0\}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ defined by

$$X_t(\omega) = \begin{cases} \sum_j x_j \, \mathbf{1}_{[0,t]}(t_j), & \text{if } \ \omega = \big((t_1, x_1), (t_2, x_2), \dots \big), \\ 0, & \text{if } \ \omega = \alpha, \end{cases}$$

is a compound Poisson process with Lévy measure λQ, and the σ-field $\mathcal{F}^X$, generated by this process, is exactly $\mathcal{F}_{\text{sym}}$.

A canonical space for a pure jump Lévy process I

Now we build a canonical pure jump Lévy process with Lévy triplet $(0, 0, \nu)$. Let $\{\varepsilon_k,\ k \geq 0\}$ a sequence of positive numbers, strictly decreasing to 0 and with $\varepsilon_0 = \infty$ and $\varepsilon_1 = 1$. Let

$$S_k = \{x \in \mathbb{R} : \varepsilon_k < |x| \leq \varepsilon_{k-1}\}.$$

Recall that since ν is a Lévy measure, $\nu(S_k) < \infty,\ \forall k$.

For each $k \geq 1$ construct the canonical compound Poisson process

$$((\Omega^{(k)}, \mathcal{F}^{(k)}_{\text{sym}}, P^{(k)}), \{X^{(k)}_t,\ t \geq 0\})$$

corresponding to the intensity $\lambda_k = \nu(S_k)$ and highs given by the law $Q_k := \nu \mathbf{1}_{S_k} / \nu(S_k)$ supported by S_k.

A canonical space for a pure jump Lévy process II

Now consider the product probability space

$$(\Omega, \mathcal{F}, P) = \bigotimes_{k\geq 1} (\Omega^{(k)}, \mathcal{F}^{(k)}_{\text{sym}}, P^{(k)}).$$

For $\omega = (\omega^{(k)})_{k\geq 1} \in \Omega$, define

$$X_t(\omega) = \lim_n \sum_{k=2}^{n} \left(X_t^{(k)}(\omega^{(k)}) - t \int_{S_k} x\,\nu(dx)\right) + X_t^{(1)}(\omega^{(1)}).$$

The existence for almost all ω of this limit is proved exactly as the Itô-Lévy representation, which gives the convergence a.s., uniform on $t \in [0, T]$, for any $T > 0$, of an equivalent sequence.

Then it is straigtforward, computing characteristic functions, that $X = \{X_t,\ t \geq 0\}$ is a cadlag Lévy process with Lévy triplet $(0, 0, \nu)$.

A CANONICAL SPACE FOR A LEVY PROCESS.

For a general Lévy process we consider:

- The canonical Brownian space $(\Omega_W, \mathcal{F}_W, \mathbb{P}_W, \{\overline{W}_t,\ t \geq 0\})$.
- The canonical pure jump Lévy space $(\Omega_J, \mathcal{F}_J, \mathbb{P}_J, \{\overline{J}_t, t \in \mathbb{R}_+\})$.

And then we define

$$(\Omega_W \times \Omega_J, \mathcal{F}_W \otimes \mathcal{F}_J, \mathbb{P}_W \otimes \mathbb{P}_J)$$

with $W_t(\omega, \omega') := \overline{W}_t(\omega)$ and $J_t(\omega, \omega') := \overline{J}_t(\omega')$.

Then, $X_t = \gamma\, t + \sigma W_t + J_t$ is the canonical Lévy process.

PROBABILISTIC INTERPRETATION OF THE OPERATOR $D_{t,0}$

We are going to see that $D_{t,0}$ turns to be the derivative with respect to the Brownian part of X and that the usual rules of classical Malliavin Calculus apply

Recall that we have the isometry

$$L^2(\Omega_W \times \Omega_J) \simeq L^2(\Omega_W; L^2(\Omega_J)),$$

and then we can apply the theory of Malliavin calculus for Hilbert space valued random variables as it is developped for example in Nualart (1995).

Let D^W the classical Malliavin derivative. Recall that its domain $\mathrm{Dom}\, D^W$ is the closure of the set $\mathcal{S}^W$ of smooth random variables, with respect to the norm

$$\|F\|_W = \Big(E[F^2] + E\big[\int_0^\infty (D_t F)^2\, dt\big]\Big)^{1/2}.$$

This domain is a Hilbert space with the scalar product

$$\langle F, G\rangle_W = E[FG] + E\big[\int_0^\infty D_t F\, D_t G\, dt\big].$$

Given a real separable Hilbert space $\mathcal{H}$, we can extend the notion of derivative for $\mathcal{H}$-valued random variables.

Specifically, let $\mathcal{S}^{W,\mathcal{H}}$ the set of $\mathcal{H}$-valued smooth random variables of the form $F = \sum_{i=1}^{n} G_i H_i$, where $G_i \in \mathcal{S}^W$ and $H_i \in \mathcal{H}$. Define $D_t^{W^*} F = \sum_{i=1}^{n} D_t G_i \otimes H_i$ and let $\mathrm{Dom}\, D^{W^*}$ be the completion of $\mathcal{S}^{W,\mathcal{H}}$ with respect to the norm

$$\|F\|_{W,\mathcal{H}} = \Big(E\big[\|F\|_{\mathcal{H}}^2\big] + E\big[\int_{\mathbb{R}_+} \|D_t^{\mathcal{H}} F\|_{\mathcal{H}}^2\, dt\big]\Big)^{1/2}.$$

Then

$$\mathrm{Dom}\, D^{W^*} \simeq \mathrm{Dom}\, D^W \otimes \mathcal{H}.$$

In the particular case of $\mathcal{H} = L^2(\Omega')$, for a certain probability space $(\Omega', \mathcal{F}', \mathbb{P}')$, such that $L^2(\Omega')$ becomes separable, we have,

$$\mathrm{Dom}\, D^{W^*} \simeq \mathrm{Dom}\, D^W \otimes L^2(\Omega') \simeq L^2(\Omega'; \mathrm{Dom}\, D^W).$$

As a consequence, if $F \in L^2(\Omega \times \Omega')$ such that for all $\omega' \in \Omega'$, $\mathbb{P}'$-a.s., $F(\cdot, \omega') \in \mathrm{Dom}\, D^W$, then $F \in \mathrm{Dom}\, D^{W^*}$ and

$$D_t^{W^*} F(\omega, \omega') = D_t^W F(\cdot, \omega')(\omega), \ell \otimes \mathbb{P} \otimes \mathbb{P}' - \text{a.e.}$$

As a corollary the following chain rule can be proved:

Let $F \in L^2(\Omega \times \Omega')$ of the form

$$F(\omega, \omega') = f(Z(\omega), Z'(\omega')),$$

where $f(x, y)$ is differentiable in x, with bounded partial derivative, and $Z \in \mathrm{Dom}\, D^W$. Then $F \in \mathrm{Dom}\, D^{W^*}$ and

$$D_t^{W^*} F = \frac{\partial f}{\partial x}(Z, Z')\, D_t^W Z.$$

In our particular context, $L^2(\Omega') := L^2(\Omega_J)$, which is a separable Hilbert space. Then

$$L^2(\Omega_W \times \Omega_J) \simeq L^2(\Omega_W; L^2(\Omega_J)),$$

and therefore we can compute both $D_{t,0}F$ and $D_t^{W^*}F$, and to obtain

$$\mathrm{Dom}\, D^{W^*} \subset \mathrm{Dom}\, D^0,$$

and for $F \in \mathrm{Dom}\, D^{W^*}$,

$$D_{t,0}F = \frac{1}{\sigma} D_t^{W^*} F.$$

This gives the probabilistic interpretation of $D_{t,0}$.

In this particular case, the chain rule establish:

Let $F = f(Z, Z') \in L^2(\Omega_W \times \Omega_J)$ with $Z \in \mathrm{Dom}\, D^W$ and $Z' \in L^2(\Omega_J)$, and $f(x, y)$ is a continuously differentiable function with bounded partial derivatives in the variable x. Then $F \in \mathrm{Dom}\, D^0$ and

$$D_{t,0}F = \frac{1}{\sigma} \frac{\partial f}{\partial x}(Z, Z')\, D_t^W Z.$$

This chain rule has been extended in Petrou (2008) in the following way:

If $F = f(Z)$ with $Z \in \mathrm{Dom}\, D^{W*}$ and f in $C_b^1(\mathbb{R})$, then

$$F \in \mathrm{Dom}\, D^{W*}$$

and

$$D_t^{W*} F = f'(Z)\, D_t^{W*} Z.$$

PROBABILISTIC INTERPRETATION OF $D_{t,x}$ FOR $x \neq 0$.

Consider now a pure jump Lévy process J with Lévy measure ν. Given $\omega \in \Omega^J$ and $z = (t,x) \in \mathbb{R}_+ \times \mathbb{R}_0$, we introduce in ω a jump of size x at instant t, and call the new element

$$\omega_z = \big((t_1, x_1), (t_2, x_2), \ldots, (t, x), \ldots\big).$$

For a $\mathcal{F}^J$-random variable F, we define the transformation

$$(T_z F)(\omega) := F(\omega_z),$$

and the application

$$TF\colon \mathbb{R}_+ \times \mathbb{R}_0 \times \Omega \longrightarrow \mathbb{R},$$

that applies (z,ω) to $F(\omega_z)$ is $\mathcal{B}(\mathbb{R}_+ \times \mathbb{R}_0) \otimes \mathcal{F}^J$ measurable and if $F = 0$, $P-$almost surely, then $TF = 0$, $\ell \otimes \nu \otimes P$ a.e..

Now we can define the increment quotient operator

$$\Psi_{t,x}F(\omega) := \frac{(T_{t,x}F)(\omega) - F(\omega)}{x}.$$

Thanks to the results of above $\Psi_{t,x}$ is a measurable operator from $L^0(\Omega^J)$ to $L^0(\mathbb{R}_+ \times \mathbb{R}_0 \times \Omega^J)$.

If $F, G \in L^0(\Omega^J)$, the operator have the following properties:

1. $\Psi(F+G) = \Psi(F) + \Psi(G)$.
2. $\Psi_{t,x}(F\,G) = G\,\Psi_{t,x}F + F\,\Psi_{t,x}G + x\,\Psi_{t,x}(F)\,\Psi_{t,x}(G)$.
3. Ψ is a closed operator in $L^2(\Omega^J)$.

Using the same ideas as in Nualart and Vives (1995), given $F \in L^2(\Omega^J)$, we have

$$F \in \mathrm{Dom}\, D^J \iff \Psi F \in L^2(\mathbb{R}_+ \times \mathbb{R}_0 \times \Omega^J),$$

and in this case

$$D_{t,x} F = \Psi F, \ \mu \otimes P - \text{a.e.}$$

This gives the probabilistic interpretation of $D_{t,x}$ for $x \neq 0$.

In the general case, consider the canonical space

$$(\Omega_W \times \Omega_J, \mathcal{F}_W \otimes \mathcal{F}_J, \mathbb{P}_W \otimes \mathbb{P}_J).$$

Given $z = (t,x) \in \mathbb{R}_+ \times \mathbb{R}_0$, for $\omega = (\omega^W, \omega^J) \in \Omega_W \times \Omega_J$ define $\omega_z = (\omega^W, \omega_z^J)$, and for a random variable $F \in L^0(\Omega_W \times \Omega_J)$ let $(T_z^* F)(\omega) := F(\omega_z)$. Define also the operator

$$\Psi^*_{t,x} F := \frac{F(\omega_{t,x}) - F(\omega)}{x}.$$

Then, for $F \in L^2(\Omega)$ we have that $F \in \mathrm{Dom}\, D$ if and only if $F \in \mathrm{Dom}\, D^{W*}$ and $\Psi^* F \in L^2(\Omega \times [0,\infty) \times \mathbb{R}_0)$, and in this case,

$$D_{t,x} F = \mathbf{1}_{\{\sigma>0\}} \mathbf{1}_{\{0\}}(x) \frac{1}{\sigma} D_t^{W*} F + \mathbf{1}_{\mathbb{R}_0}(x) \Psi^*_{t,x} F.$$

PROBABILISTIC INTERPRETATION OF δ

From now on, fix a finite time $T > 0$ and consider the process $\{X_t, \, t \in [0, T]\}$. The independent random measure M is restricted to $[0, T] \times \mathbb{R}$. Assume also $\int_{\mathbb{R}} x^2 d\nu(x) < \infty$.

Following Applebaum (2004), the random measure M, with the filtration $\left\{\mathcal{F}_t^X, \, t \in [0, T]\right\}$, induces a martingale-valued measure and allows to define an stochastic integral.

Let u be a predictable process such that $E \iint_{[0,T]\times\mathbb{R}} u^2(z)\,\mu(dz) < \infty$. We can define a stochastic integral $\iint_{[0,T]\times\mathbb{R}} u(z)\,dM_z$ such that for u and v square integrable predictable processes we have

$$E\Big[\iint_{[0,T]\times\mathbb{R}} u(z)\,dM_z\cdot\iint_{[0,T]\times\mathbb{R}} v(z)\,dM_z\Big] = E\Big[\iint_{[0,T]\times\mathbb{R}} u(z)v(z)\,d\mu(z)\Big]$$

An explicit expression for the integral $\iint_{[0,T]\times\mathbb{R}} u(z)\,dM_z$ is given by

$$\iint_{[0,T]\times\mathbb{R}} u(z)\,dM_z = \sigma\int_0^T u(t,0)\,dW_t + \iint_{[0,T]\times\mathbb{R}_0} xu(t,x)\,d\widetilde{N}(t,x).$$

As in the Brownian case, the Skorohod integral restricted to predictable processes coincides with the integral respect to the random measure M.

From Privault (1997) and Di Nunno et al. (2004) we have that if $u \in L^2([0,T] \times \mathbb{R} \times \Omega)$ is predictable, then $u \in \operatorname{Dom} \delta$ and

$$\delta(u) = \iint_{[0,T]\times\mathbb{R}} u(z)\, dM_z.$$

In fact, if δ^0 is the dual operator of $D_{t,0}$ and δ^J is the dual operator of $D_{t,x}$ for $x \neq 0$, we have

$$\delta(u) = \sigma\delta^W(u_{\cdot,0}) + \delta^J(u\mathbf{1}_{\mathbb{R}_0}(x)).$$

In particular δ^0 coincides with $\sigma\delta^W$ and δ^J coincides with the path by path integral with respect to $x\widetilde{N}$ over predictable processes.

Lemma 1

Next result will play a key role in the first application:

Let $F \in \mathrm{Dom}\, D$ be a bounded random variable and $u \in \mathrm{Dom}\, \delta$ such that

$$E \int_{[0,T]\times\mathbb{R}} (u(t,x)(F + xD_{t,x}F))^2 \mu(dt,dx) < \infty.$$

Then $u(t,x)(F + xD_{t,x}F) \in \mathrm{Dom}\, \delta$ if and only if

$$F\delta(u) - \int_{[0,T]\times\mathbb{R}} u(t,x) D_{t,x} F \mu(dt,dx) \in L^2(\Omega).$$

In this case

$$\delta(Fu) = F\delta(u) - \delta(xuDF) - \int_{[0,T]\times\mathbb{R}} u(t,x) D_{t,x} F \mu(dt,dx).$$

LEMMA 2

Under the Løkka and Petrou approach we have the following different version of the previous formula, that we use in the second application:

Let $F \in \mathrm{Dom}\,\bar{D}$ be a bounded random variable and $u \in \mathrm{Dom}\,\bar{\delta}$ such that

$$E\int_{[0,T]\times\mathbb{R}}(u(t,x)(F+\bar{D}_{t,x}F\mathbf{1}_{\{x\neq 0\}}))^2\bar{\mu}(dt,dx)<\infty.$$

Then $u(t,x)(F+\bar{D}_{t,x}F\mathbf{1}_{\{x\neq 0\}}) \in Dom\bar{\delta}$ if and only if

$$F\bar{\delta}(u)-\int_{[0,T]\times\mathbb{R}}u(t,x)D_{t,x}F\bar{\mu}(dt,dx)\in L^2(\Omega).$$

In this case

$$\bar{\delta}(Fu)=F\bar{\delta}(u)-\bar{\delta}(u\bar{D}F\mathbf{1}_{\{x\neq 0\}})-\int_{[0,T]\times\mathbb{R}}u(t,x)\bar{D}_{t,x}F\bar{\mu}(dt,dx).$$

An anticipating Itô formula

In Alòs, Léon and Vives (2008) we use the techniques presented before to obtain an anticipative version of the Itô formula for Lévy processes, where the coefficients are assumed to be in the domain of the gradient operator in the *future sense*.

This domain, introduced on the Wiener space by Alòs and Nualart (1998), includes the family of all adapted and square-integrable processes.

Therefore, this Itô formula is not only an extension of the usual adapted formula for Lévy processes, but also an extension of the anticipative version on the Wiener space.

Notation and Remarks

This application is based, technically, in the Lemma 1 presented before.

We define $L_a^2(\Omega \times [0,T])$ as the space of adapted processes such that $E\int_0^T u_s^2 ds < \infty$.

we define $\mathcal{L}_a^2(\Omega \times [0,T])$ as the space of adapted processes such that $\int_0^T |u_s|^2 ds < \infty, \quad a.s.$

Itô formula for a Lévy process I

We are interested in the semimartingale

$$X_t = X_0 + \int_0^t u_s dW_s + \int_0^t v_s ds$$
$$+ \int_0^t \int_{|y|>1} z_1(s-, y) y N(ds, dy) + \int_0^t \int_{|y|\leq 1} z_2(s-, y) y \tilde{N}(ds, dy)$$

where u and $z_2(s-, y)y$ belong to $\mathcal{L}_a^2$ and v and $z_1(s-, y)y$ belong to $\mathcal{L}_a^1$.
Observe that if $X_0 = 0$, u is a positive constant, v is a real constant and z_1 and z_2 are equal to 1, we have a generic Lévy process.

Itô formula for a Lévy process II

In this case we have

$$\begin{aligned} F(X_t) &= F(X_0) + \int_0^t F'(X_{s-}) u_s dW_s \\ &+ \int_0^t F'(X_{s-}) v_s ds + \frac{1}{2} \int_0^t F''(X_{s-}) u_s^2 ds \\ &+ \int_0^t \int_{|y|>1} [F(X_s) - F(X_{s-})] N(ds, dy) \\ &+ \int_0^t \int_{|y|\le 1} [F(X_s) - F(X_{s-}) - F'(X_{s-}) z_2(s-, y) y] N(ds, dy) \\ &+ \int_0^t \int_{|y|\le 1} F'(X_{s-}) z_2(s-, y) y \tilde{N}(ds, dy) \end{aligned}$$

Beyond Itô calculus

- Our purpose is to obtain an analogous formula changing Itô stochastic integrals by Skorohod versions, that is, an anticipating version of this formula.
- Remark that if u, v, z_1 and z_2 are anticipating processes, the Itô integral with respect to W is not defined, so we need the Skorohod extension. Moreover, the integrals with respect $\tilde{N}$ are well defined path by path, but they are not zero expectation integrals, so we are also interested in an Skorohod type version for this case.
- Coefficients will be assumed to be in the domain of the gradient operator in the future sense. So, this application includes also the Lévy extension of the corresponding domains in the Wiener case.

Itô formula for the Skorohod integral on the Wiener space

Assume

$$X_t = X_0 + \delta^W(u1\!\!1_{[0,t]}) + \int_0^t v_s ds$$

with

- $X_0 \in (\mathrm{Dom} D^W)_{loc}$
- $u \in (\mathbb{L}_W^F \cap L^\infty(\Omega, L^2[0,T]))_{loc}$ and the indefinite Skorohod integral is continuous.
- $v \in \mathbb{L}_{W,loc}^{1,2,f}$
- $F \in C^2(\mathbb{R})$.

Itô formula for the Skorohod integral on the Wiener space II

Then, $F'(X_s)u_s \mathbb{1}_{[0,t]}(s)$ belongs to $(Dom\delta^W)_{loc}$ and

$$
\begin{aligned}
F(X_t) &= F(X_0) + \delta^W(F'(X_s)u_s \mathbb{1}_{[0,t]}(s)) \\
&+ \int_0^t F'(X_s)v_s ds + \frac{1}{2}\int_0^t F''(X_s)u_s^2 ds \\
&+ \int_0^t F''(X_s)(D^{W,-}X)_s u_s ds.
\end{aligned}
$$

This formula generalizes completely the classical Itô formula.

Sobolev spaces on the Lévy space

We can extend the definitions of $\mathbb{L}_W^{1,2,f}$, $\mathbb{L}_{W,-}^{1,2,f}$ and $\mathbb{L}_W^F$ to the spaces $\mathbb{L}^{1,2,f}$, $\mathbb{L}_-^{1,2,f}$ and $\mathbb{L}^F$ in the Lévy setting.

Let $\mathcal{S}_T$ the family of smooth processes of the form

$$u(\cdot) = \sum_{j=1}^{n} F_j h_j(\cdot),$$

where F_j is a smooth random variable and $h : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a bounded function.

THE SPACE $\mathbb{L}^{1,2,f}$

We define $\mathbb{L}^{1,2,f}$ as the closure of $\mathcal{S}_T$ with respect to the seminorm

$$\begin{aligned} ||u||^2_{1,2,f} &= E\int_{[0,T]\times\mathbb{R}} u(t,x)^2\mu(dt,dx) \\ &+ E\int_{\Delta_1^T}(D_{s,y}u(t,x))^2\mu(ds,dy)\mu(dt,dx), \end{aligned}$$

where

$$\Delta_1^T = \Big\{((s,y),(t,x)) \in ([0,T]\times\mathbb{R})^2 : s \geq t\Big\}.$$

THE SPACE $\mathbb{L}_-^{1,2,f}$

A random field $u = \{u(s,y) : (s,y) \in [0,T] \times \mathbb{R}\}$ in $\mathbb{L}^{1,2,f}$ belongs to the space $\mathbb{L}_-^{1,2,f}$ if there exists $D^- u$ in $L^2(\Omega \times [0,T] \times \mathbb{R})$ such that

$$\int_0^T \int_{\mathbb{R}} \sup_{(s-\frac{1}{n})\vee 0 \le r < s, y \le x \le y + \frac{1}{n}} E[|D_{s,y} u(r,x) - D^- u(s,y)|^2] \mu(ds,dy)$$

converges to zero.

THE SPACE $\mathbb{L}^F$

The space $\mathbb{L}^F$ is the closure of $\mathcal{S}_T$ with respect to the norm

$$||u||_F^2 = ||u||_{1,2,f}^2 + E\int_{\Delta_2^T}(D_{r,x}D_{s,y}u(t,z))^2 d\mu(r,x)d\mu(s,y)d\mu(t,z),$$

with $\Delta_2^T = \{((r,x),(s,y),(t,z)) \in ([0,T]\times\mathbb{R})^3 : r\vee s \geq t\}$.
If $u \in \mathbb{L}_F$ we have $u \in Dom\delta$ and

$$E[\delta(u)^2] \leq 2||u||_F^2.$$

The proof follows from the fact that δ is closed, so we can restrict to processes in $\mathcal{S}_T$ and then to use duality relation and Lemma 1.

Relationship between Skorohod and path by path integrals

Let $z = \{z(s,x) : (s,x) \in [0,T] \times \mathbb{R}\}$ be a measurable random field such that:

- If $s_n \uparrow s$ in $[0,T]$ and $y_m \to y$ on $\mathbb{R}_0$ the limit $z(s-,y) = \lim_{n,m\to\infty} z(s_n, y_m)$ is well–defined and belongs to $\mathbb{L}_-^{1,2,f}$.
- The random fields $z(s-,y)$ and $yD^- z(s-,y)$ belongs to $\mathbb{L}^F$.
- The random field $z(s-,y)y$ is pathwise integrable with respect to $\tilde{N}$.

THEOREM 1

Then, for any interval $(a, b]$ or (a, ∞) in $(0, \infty)$ we have

$$\int_0^t \int_{\{a<|y|\leq b\}} z(s-, y) y \tilde{N}(ds, dy)$$

$$= \delta((z(s-, y) + yD^- z(s-, y)) \mathbb{1}_{\{a<|y|\leq b\}} \mathbb{1}_{[0,t]}(s))$$

$$+ \int_0^t \int_{\{a<|y|\leq b\}} D^- z(s-, y) \mu(ds, dy), \quad t \in [0, T].$$

PROOF

The proof follows from the fact that the definition of the space $\mathbb{L}^{1,2,f}$ implies that there exists a sequence of smooth processes $z^{(m)}$ that converges to z in the norm of $\mathbb{L}^{1,2,f}$.

Then, we can apply Lemma 1 to $z^{(m)}$, take limits, use the fact that the L^2-norm of δ is controlled by the norm on $\mathbb{L}^F$ and to use the dominated convergence theorem.

The Itô formula I

we consider the process

$$\begin{aligned} X_t &= X_0 + \delta^W(u \mathbb{1}_{[0,t]}) + \int_0^t v_s ds \\ &+ \int_0^t \int_{\{|x|>1\}} z_1(s-,x) x N(ds,dx) \\ &+ \int_0^t \int_{\{0<|x|\leq 1\}} z_2(s-,x) x \tilde{N}(ds,dx), \quad t \in [0,T]. \end{aligned}$$

THE ITÔ FORMULA II

Assume the following hypotheses

- $X_0 \in \mathrm{DomD}$.
- $u \in \mathbb{L}^F$, $\delta^W(u 1\!\!1_{[0,t]})$ has continuous paths and $\int_0^T u_s^2 ds$ is a.s. bounded by a constant.
- $v \in \mathbb{L}^{1,2,f}$ and $\int_0^T v_s^2 ds$ is a.s. bounded by a constant.
- z_1 and z_2 are bounded and satisfies the conditions of Theorem 1 on $(1, \infty)$ and $(0, 1]$ respectively. Moreover, $D^- z_2 \in \mathbb{L}^{1,2,f}$

The Itô formula III

Assume X satisfies the last hypothesis and $F \in C^2(\mathbb{R})$. Then, the process

$$F'(X_{s-})(u_s 1\!\!1_{\{y=0\}} + z_2(s-,y) 1\!\!1_{\{0<|y|\leq 1\}}) 1\!\!1_{[0,t]}(s)$$

$$+D^-(z_2(s-,y)F'(X_{s-}))(s,y)y 1\!\!1_{\{0<|y|\leq 1\}} 1\!\!1_{[0,t]}(s)$$

belongs to $Dom\delta$ and

Itô formula IV

$$
\begin{aligned}
& F(X_t) - F(X_0) \\
= \;& \delta((F'(X_{s-})(u_s 1\!\!1_{\{y=0\}} + z_2(s-,y) 1\!\!1_{\{0<|y|\leq 1\}}) \\
+ \;& y 1\!\!1_{\{0<|y|\leq 1\}} D^-_{(s,y)}(z_2(s-,y)F'(X_{s-}))) 1\!\!1_{[0,t]}(s)) \\
+ \;& \frac{1}{2}\int_0^t F''(X_s)u_s^2 ds + \int_0^t F'(X_s)v_s ds + \int_0^t F''(X_s)D^-_{(s,0)}X_s u_s ds \\
+ \;& \int_0^t \int_{\{0<|y|\leq 1\}} D^-_{(s,y)}F'(X_{s-})z_2(s,y)\mu(ds,dy) \\
+ \;& \int_0^t \int_{0<|y|\leq 1} [F(X_s) - F(X_{s-}) - F'(X_{s-})z_2(s-,y)y] N(ds,dy) \\
+ \;& \int_0^t \int_{\{|y|>1\}} (F(X_s) - F(X_{s-}))N(ds,dy)
\end{aligned}
$$

A Hull and White formula for SV-jump-diffusion models

In this section we apply the same techniques to obtain a Hull and White formula for a very general jump-diffusion price model with stochastic volatility, where the volatility does not need to be neither a diffusion nor a Markov process.

NOTATION

- Let be $S = \{S_t, t \geq 0\}$ a price process.
- Let $\{\mathcal{F}_t, t \geq 0\}$ be the information flow given by the process.
- Let $T > 0$ be a finite horizon.
- Let $r > 0$ be the fixed interest rate.
- Let $X_t = \log S_t$ be the log price process.

THE MODEL I

We assume that the log-price, under a market chosen risk-neutral probability $\mathbb{Q}$, follows the dynamics:

$$X_t = x + (r - \lambda k)t - \frac{1}{2}\int_0^t \sigma_s^2 ds + \int_0^t \sigma_s(\rho dW_s + \sqrt{1-\rho^2}dB_s) + Z_t.$$

where x is the current log-price, W and B are independent standard Brownian motions, $\rho \in [-1, 1]$ and Z is a compound Poisson process with intensity λ, Lévy measure ν, independent of W and B, and with $k = \frac{1}{\lambda}\int_{\mathbb{R}}(e^y - 1)\nu(dy)$.

THE MODEL II

The volatility process σ is assumed to be a square-integrable stochastic process with right-continuous trajectories, bounded below by a positive constant and adapted to the filtration generated by W and Z.

REMARKS 1

- This model is a generalization of the Bates model (1996) in the sense that we do not assume any concrete dynamics for the stochastic volatility. In fact we change the dynamics of σ by Malliavin calculus conditions.
- If we assume no jumps ($\nu = 0$ and $\lambda = 0$), we have a generalized Heston model (1993) in the same sense as before.
- If in addition $\rho = 0$ we have a generalization of different stochastic volatility models as Hull - White (1987), Scott (1987), Stein - Stein (1991) or Ball - Roma (1994). If σ is deterministic or constant we have the Osborne-Samuelson-Black-Scholes model.

REMARKS 2

- It is very well known that a major problem of Blak-Scholes model is the empirical fact that volatility is not constant across diferent strike prices and times to maturity.
- Stochastic volatility models pursue the goal to replicate price surfaces of plain vanilla options (depending on times to maturity and strikes) given by derivative markets or vanilla desks.
- Stochastic volatility is a non observable process, so it is not easy to model. This is a justification for trying to assume minimal conditions on the process σ.

THE HULL AND WHITE FORMULA I

Let be H_T the profit profile of a financial derivative. In particular it is a $\mathcal{F}_T-$measurable functional of S, and so, of W, B and Z.

The Hull and White formula is the pricing formula for H_T:

$$V_t = e^{-r(T-t)}\mathbb{E}_t^{\mathbb{Q}}(H_T),$$

where E_t denotes the conditional expectation with respect to $\mathcal{F}_t$.

Observe that the practical situation requires to compute $V_0 = e^{-rT}\mathbb{E}^{\mathbb{Q}}(H_T)$.

To fix ideas we will concentrate on the case of a plain vanilla call, that is, $H = (S_T - K)^+$.

THE HULL AND WHITE FORMULA II

Under the Black-Scholes model, the Hull and White formula for $t = 0$ is nothing more that the Black-Scholes formula.

It easy to see that under a stochastic volatilty model with no correlation and no jumps, the formula is the Black-Scholes formula changing σ by the average of future volatilities $\bar{\sigma}_t$ where

$$\bar{\sigma}_t^2 := \frac{1}{T-t} \int_t^T \sigma_s^2 ds.$$

Why Malliavin calculus?

- The dependence of options prices in SV-models on the average future volatility process $\bar{\sigma}$, that is a non-adapted process, suggests the introduction of Malliavin calculus as a natural tool to deal with this type of processes.
- Moreover, Malliavin calculus allows to go beyond the Markovian setting relaxing the conditions on the volatility process.
- This is the basis of the papers: Alòs (2006), for the correlated but no jumps case, Alòs, León and Vives (2007), for the correlated case with jumps on the price process and Alòs, León, Pontier and Vives (2008) for the correlated case with jumps both on price process and on the volatility process.

The Hull and White formula III

For the correlated case plus jumps, but with jumps only on the price process and not on the volatility, that is assuming only $\sigma \in \mathcal{F}^W$, we have the following HW pricing formula:

$$V_t = \mathbb{E}_t(BS(t, X_t, \bar{\sigma}_t))$$

$$+\frac{\rho}{2}\mathbb{E}_t(\int_t^T e^{-r(s-t)}(\partial_{xxx} - \partial_{xx})BS(s, X_s, \bar{\sigma}_s)\Lambda_s ds)$$

$$+\mathbb{E}_t(\int_t^T \int_{\mathbb{R}} e^{-r(s-t)}(BS(s, X_s + y, \bar{\sigma}_s) - BS(s, X_s, \bar{\sigma}_s))\nu(dy)ds)$$

$$-\mathbb{E}_t(\int_t^T \int_{\mathbb{R}} e^{-r(s-t)}\partial_x BS(s, X_s, \bar{\sigma}_s)(e^y - 1)\nu(dy)ds).$$

REMARKS

- This formula shows that the European call price is a sum of the price when the model has no jumps and no correlation plus two terms, one describing the impact of the correlation and the other the impact of jumps. Hence this formula will be a useful tool to compare the effect of correlation and jumps.
- We have needed Brownian Malliavin calculus for dealing with the process σ, adapted to the Brownian filtration. If we want to deal with σ adapted to the filtration generated by W and Z we will need Lévy Malliavin calculus.

Malliavin calculus tools I

In our case, we suppose $E[\int_0^T \sigma_s^2 ds] < \infty$ and $\int_{\mathbb{R}} x^2 \nu(dx) < \infty$. Moreover the Lévy measure ν is finite. We will use the Løkka and Petrou approach.

In the remaining of this paper, we will denote $D^N_{t,x} := T_{t,x} - Id$, of course only defined on $[0,T] \times \mathbb{R}_0$. Observe that we have

$$\bar{D}_{t,x} = 1\!\!1_{\{\sigma>0\}} 1\!\!1_{\{0\}}(x) \frac{1}{\sigma} D_t^W + 1\!\!1_{\mathbb{R}_0}(x) D^N_{t,x}.$$

and

$$D^N_{t,x} F = F D^N_{t,x} G + G D^N_{t,x} F + D^N_{t,x} F D^N_{t,x} G.$$

Malliavin calculus tools II

Next proposition is the main technical result and is the basis to give the relation between $\bar{\delta}$ and the path by path integral with respect to N.
We need to introduce some spaces. We define

$$\mathbb{L}^{1,2} := L^2([0,T] \times \mathbb{R}; \mathrm{Dom}\bar{D}).$$

Remark that if $u = \{u(s,y) : (s,y) \in [0,T] \times \mathbb{R}\}$ is a random field of $\mathbb{L}^{1,2}$ we have, in particular, that u and $\bar{D}u$ are in $L^2(\mathbb{P} \otimes \bar{\mu})$ and $L^2(\mathbb{P} \otimes \bar{\mu} \otimes \bar{\mu})$ respectively.

Malliavin calculus tools III

Moreover we define $\mathbb{L}_{-}^{1,2}$ as the subset of $\mathbb{L}^{1,2}$ of random fields u such that the following $\mathbb{P} \otimes \bar{\mu}$-a.s. left-limits exists and belong to $L^2(\mathbb{P} \otimes \bar{\mu})$:

$$u^{-}(s,y) = \lim_{r \uparrow s, x \uparrow y} u(r,x),$$

$$D^{-}u(s,y) = \lim_{r \uparrow s, x \uparrow y} \bar{D}_{s,y} u(r,x).$$

THEOREM 2 I

Let $u \in \mathbb{L}_{-}^{1,2}$. Let be $T^{-}u := u^{-} + D^{-}u$. Assume

$$\int_0^T \int_{\mathbb{R}_0} |u^{-}(s,x)| N(ds,dx) \in L^2(\Omega),$$

where $\int_0^T \int_{\mathbb{R}_0} u(s,x) N(ds,dx)$ is the classical path-by-path integral defined by $\sum_{\Delta Z_t \neq 0} u(t, \Delta Z_t)$. Then, $T^{-}u = u^{-} + D^{-}u \in Dom\bar{\delta}$, and in this case,

$$\bar{\delta}((u^{-} + D^{-}u)1\!\!1_{\mathbb{R}_0}) = \int_0^T \int_{\mathbb{R}_0} u^{-} d\tilde{N} - \int_0^T \int_{\mathbb{R}_0} D^{-}u(s,x)\nu(dx)ds.$$

Theorem 2 II

Equivalently, we have

$$\bar{\delta}(T^{-}u \cdot 1\!\!1_{\mathbb{R}_0}) = \int_0^T \int_{\mathbb{R}_0} u^{-}(s,x) N(ds,dx) - \int_0^T \int_{\mathbb{R}_0} T^{-}u(s,x)\nu(dx)ds.$$

REMARK

Observe that when u is adapted to the filtration generated by N, $D^- u = 0$ and $T^- u = u^-$. Therefore in such a case

$$\int_0^t \int_{\mathbb{R}_0} u^-(s,y)\tilde{N}(ds,dy) = \bar{\delta}(u^-(\cdot,\cdot)1\!\!1_{[0,t]\times\mathbb{R}_0}(\cdot,\cdot)).$$

That is, for predictable processes, the path by path integral with respecte to $\tilde{N}$ and the Skorohod integral are the same.

The Hull and White formula IV

In this case, the Hull and White formula is

$$V_t = E_t(BS(t, X_t, \bar{\sigma}_t))$$

$$+\frac{\rho}{2}E_t(\int_t^T e^{-r(s-t)}(\partial_{xxx} - \partial_{xx})BS(s, X_s, v_s)\Lambda_s ds)$$

$$+E_t(\int_t^T \int_{\mathbb{R}} e^{-r(s-t)}(T^- BS(s, X_{s-} + y, \bar{\sigma}_s) - T^- BS(s, X_{s-}, \bar{\sigma}_s))\nu(dy)ds)$$

$$-\lambda k E_t(\int_t^T e^{-r(s-t)}\partial_x BS(s, X_s, \bar{\sigma}_s)ds).$$

Remark I

Recall that in the case that σ only depends on the filtration generated by W, we have $T^- = Id$. Consequently, in this case, we obtain the Hull and White formula given in Alòs, León and Vives (2007) and presented before.

REMARK II

The additional term given by $T^{-}BS$ can be detailed as following:

$$T^{-}BS(s, X_{s-} + y, \bar{\sigma}_s) = BS(s, X_{s-} + y, \bar{\sigma}_s(\omega_{s,y}))$$

and

$$\bar{\sigma}_s(\omega_{s,y}) = \frac{1}{T-s}\int_s^T \hat{\sigma}_r^2 dr$$

where

$$\hat{\sigma}_r^2 = f(W_u, Z_u + y\mathbb{1}_{\{s \le u\}}, u \le r).$$