On the graphs of non-differentiable functions of the Weierstrass type

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Introduction

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Setup

We study graphs of real functions of the form

$$f : \mathbb{R} \to \mathbb{R}, \qquad f(x) = \sum_{n=0}^{\infty} a_n \, g(b_n x + \theta_n),$$

where:

- $g : \mathbb{R} \to \mathbb{R}$ non-constant periodic Lipschitz function,
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We are interested in the Hausdorff, lower and upper box (Minkowski) dimension, denoted respectively by

$$\dim_H, \quad \underline{\dim}_B, \quad \overline{\dim}_B$$

of the graph of f.

Geometric case ($a_n = a^n$, $b_n = b^n$ for $0 < a < 1 < b$)

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Historical motivation — the Weierstrass function

$$W(x) = \sum_{n=0}^{\infty} a^n \cos(b^n x)$$

for $0 < a < 1 < b$, $ab \geq 1$ is a continuous, nowhere differentiable function (Weierstrass 1872, Hardy 1916).

Relation to dynamical systems

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Remark

If b is a positive integer, g is 1-periodic smooth function, then the graph of

$$f(x) = \sum_{n=0}^{\infty} a^n g(b^n x)$$

is a compact invariant repeller for the smooth dynamical system given by

$$F : \mathbb{R}/\mathbb{Z} \times \mathbb{R} \to \mathbb{R}/\mathbb{Z} \times \mathbb{R}, \qquad F(x, y) = \left(bx \pmod 1, \frac{y - g(x)}{a} \right),$$

The system has two different positive Lyapunov exponents $\log b$, $-\log a$. The graph of f is the common boundary of the basins of attraction given by $y_n \to +\infty$, $y_n \to -\infty$, where $F^n(x, y) = (x_n, y_n)$.

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  For $g \in C^1$, $\theta_n = 0$ we have $f \in C^1$ or $\dim_B(\text{graph } f) = 2 - \alpha$, where $\alpha = -\frac{\log a}{\log b}$ is the Hölder exponent of f.

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- **[Mauldin–Williams 1986]**
  For $g \in C^1$ non-constant, $a = b^{-\alpha}$ $(0 < \alpha < 1)$, b large, we have $\dim_H(\text{graph } f) > 2 - \alpha - \frac{c}{\log b}$, c independent of b.

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- **[Przytycki, Urbański 1989]**
  For $g = \cos$ (or similar), $b = 2, 3, \ldots$ **or** g general, b large, we have $\dim_H(\text{graph } f) > 1$.

Results in the geometric case — continued

- **[Ledrappier 1992]**
  For $b = 2$, $g(x) = \mathrm{dist}(x, \mathbb{Z})$ (called "sawtooth" or Tagaki function), $1/2 < a = 2^{-\alpha} < 1$ such that Bernoulli convolution $\sum \pm a^n$ is absolutely continuous (holds for a.e. $a \in (1/2, 1)$), then $\dim_H(\text{graph } f) = 2 - \alpha$.

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- Many other works (**Bousch, Heurteaux, Hu, Lau, . . .**).

Main open problem

Determining $\dim_H(\text{graph } f)$ in the general case.

Case $b_{n+1}/b_n \longrightarrow \infty$

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- **[Besicovitch–Ursell 1937]** Let $g(x) = \text{dist}(x, \mathbb{Z})$, $a_n = b_n^{-\alpha}$ for some $\alpha \in (0, 1)$, $\theta_n = 0$.
  - If b_{n+1}/b_n increases to ∞ and $\log b_{n+1}/\log b_n \to 1$, then $\dim_H(\text{graph } f) = 2 - \alpha$.
  - If $b_n = b_1^{\beta^{n-1}}$ for $b_1 > 1$ and $\beta = \frac{(1-\alpha)(2-H)}{\alpha(H-1)}$ for $1 < H < 2 - \alpha$, then $\dim_H(\text{graph } f) = H$.

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- **[Wingren 1995]** For $g(x) = \mathrm{dist}(x, \mathbb{Z})$, $a_n = 2^{-n}$, $b_n = 2^{2^n}$, $\theta_n = 0$, we have $\dim_H(\mathrm{graph}\, f) = 2$.
- **[Liu 2001]** For $g(x) = \mathrm{dist}(x, \mathbb{Z})$, $a_n = 2^{-n(n-1)-1}$, $b_n = 2^{n(n+1)+1}$, $\theta_n = 0$, we have $\dim_H(\mathrm{graph}\, f) = 1$.

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- **[Carvalho 2011]** If g increasing (resp. decreasing) on some interval I_1 (resp. I_2) with $|g(x) - g(y)| > \delta|x - y|$ on I_1 and I_2, $\log b_{n+1}/\log b_n \to \beta$ and $-\log b_n/\log a_n \to \alpha$ for $\alpha \in (0,1)$, $\beta > 1$, then

$$\dim_H(\operatorname{graph} f) = \underline{\dim}_B(\operatorname{graph} f) = 1 + \frac{1-\alpha}{1-\alpha+\alpha\beta},$$

$$\overline{\dim}_B(\operatorname{graph} f) = 2 - \alpha.$$

Theorem (B, 2011)

Let $f(x) = \sum_{n=0}^{\infty} a_n\, g(b_n x + \theta_n)$, *where*

- $g : \mathbb{R} \to \mathbb{R}$ *periodic, Lipschitz, strictly monotone on some interval* I *with* $|g(x) - g(y)| > \delta |x - y|$ *for* $x, y \in I$ *and a constant* $\delta > 0$,
- $a_n, b_n > 0$, $\dfrac{a_{n+1}}{a_n} \to 0$, $\dfrac{b_{n+1}}{b_n} \to \infty$,
- $\theta_n \in \mathbb{R}$.

Then

$$\dim_H(\text{graph } f) = \underline{\dim}_B(\text{graph } f) = 1 + \liminf_{n\to\infty} \frac{\log^+ d_n}{\log(b_{n+1} d_n / d_{n+1})},$$

$$\overline{\dim}_B(\text{graph } f) = 1 + \limsup_{n\to\infty} \frac{\log^+ d_n}{\log b_n},$$

where

$$d_n = a_1 b_1 + \cdots + a_n b_n.$$

Remarks

- The assumptions on the function g are satisfied, if g is a periodic Lipschitz function, which is non-constant and C^1 on some interval (e.g. if g is a periodic non-constant C^1 function).
- The proof shows that $\dim_H(\operatorname{graph} f) = \dim_H(\operatorname{graph} f|_{\mathcal{I}})$ for some Cantor set $\mathcal{I} \subset \mathbb{R}$ of Lebesgue measure 0.
- The assertion on $\overline{\dim}_B(\operatorname{graph} f)$ holds under a weaker assumption: $\frac{a_{n+1}}{a_n} \to 0$ can be replaced by $a_{n+1} < \eta a_n$ for large n, where η is a sufficiently small constant depending on g (not on the sequences a_n, b_n, θ_n).

Corollary

If additionally $a_{n+1}b_{n+1} \geq a_n b_n \geq 1$ *for sufficiently large* n, *then*

$$\dim_H(\text{graph } f) = \underline{\dim}_B(\text{graph } f) = 1 + \liminf_{n\to\infty} \frac{\log(a_n b_n)}{\log(a_n b_n / a_{n+1})},$$

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In this case $\dim_B(\text{graph } f)$ *exists if and only if there exists the limit* $\gamma = \lim_{n\to\infty} \frac{\log^+ d_n}{\log b_n}$ *and then*

$$\dim_H(\text{graph } f) = \dim_B(\text{graph } f) = 1 + \gamma.$$

Corollary

Let g be as above and let

$$f(x) = \sum_{n=0}^{\infty} b_n^{-\alpha} g(b_n x + \theta_n),$$

where $0 < \alpha \le 1$, $b_n > 0$, $\frac{b_{n+1}}{b_n} \to \infty$ *and* $\theta_n \in \mathbb{R}$. *Then*

$$\dim_H(\text{graph } f) = \underline{\dim}_B(\text{graph } f) = 1 + \frac{1-\alpha}{1 - \alpha + \alpha \limsup_{n\to\infty} \frac{\log b_{n+1}}{\log b_n}},$$

$$\overline{\dim}_B(\text{graph } f) = 2 - \alpha.$$

(*Including the case* $\limsup_{n\to\infty} \frac{\log b_{n+1}}{\log b_n} = \infty$ *with the convention* $1/\infty = 0$.) *In particular, if* $\frac{\log b_{n+1}}{\log b_n} \to 1$, *then*

$$\dim_H(\text{graph } f) = \dim_B(\text{graph } f) = 2 - \alpha.$$

Corollary

For every $1 \leq H \leq B \leq 2$, *there exists a function f fulfilling the assumptions of the above theorem, such that*

$$\dim_H(\text{graph } f) = \underline{\dim}_B(\text{graph } f) = H, \qquad \overline{\dim}_B(\text{graph } f) = B.$$

Examples (g can be taken to be sin, cos, dist$(\cdot, \mathbb{Z})$ etc.)

- $1 \leq H = B < 2$ ▶ $f(x) = \sum_{n=1}^{\infty} n^{-(2-B)n} g(n^n x)$,
- $1 < H < B < 2$ ▶ $f(x) = \sum_{n=1}^{\infty} 2^{-(2-B)\beta^n} g(2^{\beta^n} x)$ for $\beta = \frac{(2-H)(B-1)}{(H-1)(2-B)}$,
- $1 = H < B < 2$ ▶ $f(x) = \sum_{n=1}^{\infty} 2^{-(2-B)n^n} g(2^{n^n} x)$,
- $H = 1,\ B = 2$ ▶ $f(x) = \sum_{n=1}^{\infty} 2^{-\frac{n^n}{\sqrt{n}}} g(2^{n^n} x)$,
- $1 < H < B = 2$ ▶ $f(x) = \sum_{n=1}^{\infty} 2^{-\frac{2-H}{e(H-1)} n^{n-1}} g(2^{n^n} x)$,
- $H = B = 2$ ▶ $f(x) = \sum_{n=1}^{\infty} n^{-\sqrt{n}} g(n^n x)$.

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with $I_{n,j} \subset I_{n-1,j'}$ for some j'.

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Then

$$b_n x + \theta_n \in I \mod 1 \qquad \text{for } x \in I_{n,j},$$

so

$$|f_n(x) - f_n(y)| > c d_n |x - y| \qquad \text{for } x, y \in I_{n,j},$$

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Hence,

$$|f(x) - f(y)| > c_1 d_n |x - y| - c_2 a_{n+1} \qquad \text{for } x, y \in I_{n,j}.$$

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Let μ be the probabilistic measure on $\mathcal{I}$, such that for every $I_{n+1,j'} \subset I_{n,j}$,

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Let ν be the lift of μ to graph $f|_{\mathcal{I}}$.
We estimate

$$\liminf_{r \to 0^+} \frac{\log \nu(Q_r(t))}{\log r}$$

where $Q_r(t)$ be the square with horizontal and vertical sides of length r centred at $(t, f(t)) \in$ graph f.