# Multifractal spectrum of generic measures and functions monotone in several variables

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## Hölder spectrum of typical monotone continuous functions

joint work with Judit Nagy

The space of monotone increasing continuous functions equipped with the sup norm is denoted by $C_{\nearrow}[0,1]$.
It is a separable, complete metric space, being a closed subspace of $C[0,1]$. If a property holds for a residual subset of $C_{\nearrow}[0,1]$ then we say that the generic (typical) function $f$ has this property.

Zamfirescu: the generic monotone continuous function, $f$ is a strictly monotone increasing singular function and its derivative equals 0 wherever it exists. Of course, $f'$ exists almost everywhere in $[0,1]$.

S. Jaffard has studied the multifractal properties of several specific continuous functions. For these functions many interesting methods, for example, wavelets, Diophantine approximation etc. were used.

It is not difficult toverify that the typical continuous function on $[0,1]$ is quite uninteresting from our point of view, it does not have multifractal Hölder properties, (it is a monofractal) in fact, it is Hölder class 0 everywhere in $[0,1]$. While the class of typical monotone continuous functions are of multifractal nature. By studying typical monotone continuous functions we study typical continuous measures on $[0,1]$.

Let $d$ be an integer greater than one.
A function $f : [0,1]^d \to \mathbb{R}$ is continuous monotone increasing in several variables (in short: MISV) if for all $i \in \{1, ..., d\}$,
the functions $f^{(i)}(t) = f(x_1, ..., x_{i-1}, t, x_{i+1}, ..., x_d)$
are continuous monotone increasing.

We set $\mathcal{M} = \{f \in C([0,1]^d) : f \text{ MISV}\}$.

The space $\mathcal{M}$ is a separable complete metric space when equipped with the supremum norm for functions, that we denote by $\|.\|$.

The multifractal properties of functions in $\mathcal{M}^1$ have been examined by Z. B. and J. Nagy in 2000.

In this talk, we study the case of higher dimensional functions $d \geq 2$.

We also investigate the level set structure of such functions.

## Hölder exponent and singularity spectrum for a locally bounded function.

**D**.: Let $f \in L^\infty([0,1]^d)$. For $h \geq 0$ and $x \in [0,1]^d$, the function $f$ belongs to $C^h_x$ if there are a polynomial $P$ of degree less than $[h]$ and a constant $C$ such that, for $x'$ close to $x$,

$$|f(x') - P(x'-x)| \leq C|x'-x|^h.$$

The pointwise Hölder exponent of $f$ at $x$ is $h_f(x) = \sup\{h \geq 0 : \ f \in C^h_x\}$.

When $h_f(x) < 1$, the pointwise Hölder exponent of $f$ at $x$ is also given by the formula

$$h_f(x) = \liminf_{x' \to x} \frac{\log |f(x') - f(x)|}{\log |x'-x|}.$$

**D**.: singularity spectrum of $f$ is defined by

$$d_f(h) = \dim_H E^h_f, \quad \text{where } E^h_f = \{x : h_f(x) = h\}.$$

$\dim_H$ = Hausdorff dimension, and $\dim \emptyset = -\infty$ by convention.

We will also use the sets $E^{h,\leq}_f = \{x : h_f(x) \leq h\} \supset E^h_f$.

The typical properties of functions in $\mathcal{M}^1$:

**T**.: (Z.B. & J. Nagy) *Consider the space of monotone continuous functions $\mathcal{M}^1$ defined on* $[0,1]$.
*(i) For every $f \in \mathcal{M}^1$, for every $h \geq 0$, one has $d_f(h) \leq \min(h,1)$.*
*(ii) There exists a residual set $\mathcal{R}_1$ in $\mathcal{M}^1$ such that for every $f \in \mathcal{R}_1$,*
$d_f(h) = h$ *for every $h \in [0,1]$, and $E_f^h = \emptyset$ if $h > 1$.*
*(iii) $\mu_f([0,1] \setminus E_f^{0,\leq}) = 0$, where $\mu_f$ is the Borel integral of $f$: $f(x) = \int_0^x d\mu_f$.*

By (i) of the above theorem, $0 = \dim_H E_f^0 = \dim_H E_f^{0,\leq}$.
(iii) shows that all the „increasing" of $f$ takes place on this set $E_f^0$ of zero Hausdorff dimension.
Since a typical monotone function is strictly monotone increasing, its level sets are points.
We deduce that heuristically, for „most" levels in the range of $f$, the corresponding points belong to the zero dimensional set $E_f^0$.

The new higher dimensional results:

An upper estimate of the singularity spectrum which is valid for arbitrary functions in $\mathcal{M}$.

**T**.: *For all* $f \in \mathcal{M}$ *and* $h \geq 0$, *we have*

$\dim_H E_f^{h,\leq} \leq \min(d-1+h, d).$

*In particular,* $d_f(h) = \dim_H(E_f^h) \leq \min(d-1+h, d)$.

One recognizes item (i) of the previous theorem in the case $d = 1$.
The next theorem shows that for $h \in [0, 1]$ the generic functions can be as bad as possible from the multifractal standpoint:

**T**.: *There exists a dense* $G_\delta$ *set* $\mathcal{R} \subset \mathcal{M}$ *such that for all* $f \in \mathcal{R}$ *we have* $d_f(h) = d - 1 + h$ *for all* $h \in [0, 1]$. *For these functions, for every* $h > 1$ *the set* $E_f^h$ *is empty.*

## Level sets of MISV functions.

We define for every $a \in \mathbb{R}$ the level set $L_f(a)$ by

$$L_f(a) = \{x \in [0,1]^d : f(x) = a\}.$$

It is easy to see that for any continuous function the Hausdorff dimension of the level sets $L_f(a)$ in the interior of the range of an $f \in \mathcal{M}$ is at least $d-1$, for every $a$.

**T**.: *There exists a dense $G_\delta$ subset $\mathcal{L}$ in $\mathcal{M}$ such that for all $f \in \mathcal{L}$ the following holds.*
*There exist a set $X_f \subset [0,1]^d$ and a set $A_f \subset (f(0,...,0), f(1,...,1)) = (m_f, M_f)$ satisfying:*

- $\dim_H X_f = d-1$, $\dim_H A_f = 0$,
- *for every $a \in (m_f, M_f)$, there is at most one point of $L_f(a)$ which does not belong to $X_f$ (in other words, $L_f(a) \cap ([0,1]^d \setminus X_f)$ contains at most one point).*
- *for every $a \in (m_f, M_f) \setminus A_f$, $L_f(a) \subset X_f$.*

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- *for every $a \in (m_f, M_f) \setminus A_f$, $L_f(a) \subset X_f$.*

In other words, $X_f$ contains Lebesgue-almost every level sets $L_f(a)$, and for those level sets $L_f(a)$ which are not entirely contained in $X_f$ (this occurs for a set of values of $a$ of Hausdorff dimension 0), exactly one point of $L_f(a)$ does not belong to $A_f$.
This entails that our function $f$ is „increasing" only on the small $d-1$ dimensional set $X_f$ which has the minimum possible dimension to contain at least one level set.
Most points in the domain of $f$ belong to $[0,1]^d \setminus X_f$, which can intersect just „very few" level sets and in no more than one point.
In particular, for all $x, x' \in [0,1]^d \setminus X_f$ (this set has full Lebesgue measure in $[0,1]$), $f(x) \neq f(x')$.

Set $\mathbb{R}^d_+ = \{(l_1, ..., l_d) : \forall i,\ l_i \geq 0\}$ and $\mathbb{R}^d_- = \{(l_1, ..., l_d) : \forall i,\ l_i \leq 0\}$.

It is well-known that generic continuous functions on $[0, 1]$ are nowhere monotone.

MISV functions are obviously monotone increasing along lines $\underline{l}t + \underline{b} = (l_1 t + b_1, ..., l_d t + b_d)$, $(t \in \mathbb{R})$ if $\underline{l} \in \mathbb{R}^d_+$ and monotone decreasing if $\underline{l} \in \mathbb{R}^d_-$.

For the generic functions in $\mathcal{M}$ one cannot say much more:

**T**.: *There exists a dense $G_\delta$ subset $\mathcal{G}$ in $\mathcal{M}$ such that for any $f \in \mathcal{G}$, if $\underline{l} = (l_1, ..., l_d) \notin \mathbb{R}^d_+ \cup \mathbb{R}^d_-$ and $\underline{b} = (b_1, ..., b_d) \in \mathbb{R}^d$, then the function $g_{\underline{l},\underline{b}}(t) = f(\underline{l}t + \underline{b})$, $t \in \mathbb{R}$ is monotone on no non-empty open subinterval on its domain.*

A two-dimensional variant of this result was Problem 6 of the annual Miklós Schweitzer competition of the János Bolyai Mathematical Society in 2010. Here we state this problem using our notation:

**Problem for the Schweitzer competition.** *Does there exist a function $f(x, y)$ which is continuous on $\mathbb{R}^2$ and such that for all $b \in \mathbb{R}$, the function $g_{l,b}(t) = f(t, lt + b)$ is strictly increasing on $\mathbb{R}$ for $l \geq 0$ and monotone on no nonempty open interval for $l < 0$?*

Generic functions in the compact-open topology of $C(\mathbb{R}^2)$ provide functions requested by the above problem, but there are not too difficult direct constructions as well.

## Typical Borel measures on $[0,1]^d$ satisfy a multifractal formalism

Let $\mathcal{M}([0,1]^d)$ be the set of probability measures on $[0,1]^d$ endowed with the weak topology (which makes $\mathcal{M}([0,1]^d)$ a compact separable space).

Let us denote by $\text{Lip}([0,1]^d)$ the set of Lipschitz functions on $[0,1]^d$ with Lipschitz constant not exceeding 1. Recall that the weak topology on $\mathcal{M}([0,1]^d)$ is induced by the following metric: if $\mu$ and $\nu$ belong to $\mathcal{M}([0,1]^d)$, we set $\varrho(\mu,\nu) = \sup\left\{\left|\int f\, d\mu - \int f\, d\nu\right| : f \in \text{Lip}([0,1]^d)\right\}.$

$\mathcal{M}([0,1]^d)$ is a separable set.

For our purpose, we specify a countable dense family of atomic measures. If $j$ is an integer greater than 1, then we set $\mathbb{Z}_j = \{0,1,\cdots,2^j-1\}^d$.

The set of finite atomic measures of the form $\displaystyle\sum_{\mathbf{k}\in\mathbb{Z}_j} r_{j,\mathbf{k}} \cdot \delta_{\mathbf{k}2^{-j}},$

where $j \in \mathbb{N}$, and $(r_{j,\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}_j}$ are (strictly) positive rational numbers such that $\displaystyle\sum_{\mathbf{k}\in\mathbb{Z}_j} r_{j,\mathbf{k}} = 1,$ forms a dense set in $\mathcal{M}([0,1]^d)$ for the weak topology.

The local regularity of a positive measure $\mu \in \mathcal{M}([0,1]^d)$ at a given $x_0 \in [0,1]$ is quantified by a *local dimension* (or a *local Hölder exponent*) $h_\mu(x_0)$, defined as $h_\mu(x_0) = \liminf_{r\to 0^+} \frac{\log \mu(B(x_0,r))}{\log r}$,

where $B(x_0,r)$ denotes the ball with center $x_0$ and radius $r$.

In geometric measure theory $h_\mu(x_0)$ is called the lower local dimension of $\mu$ at $x_0$ and is denoted by $\underline{\dim}_{\text{loc}}\mu(x_0)$.

Then the singularity spectrum of $\mu$ is the map

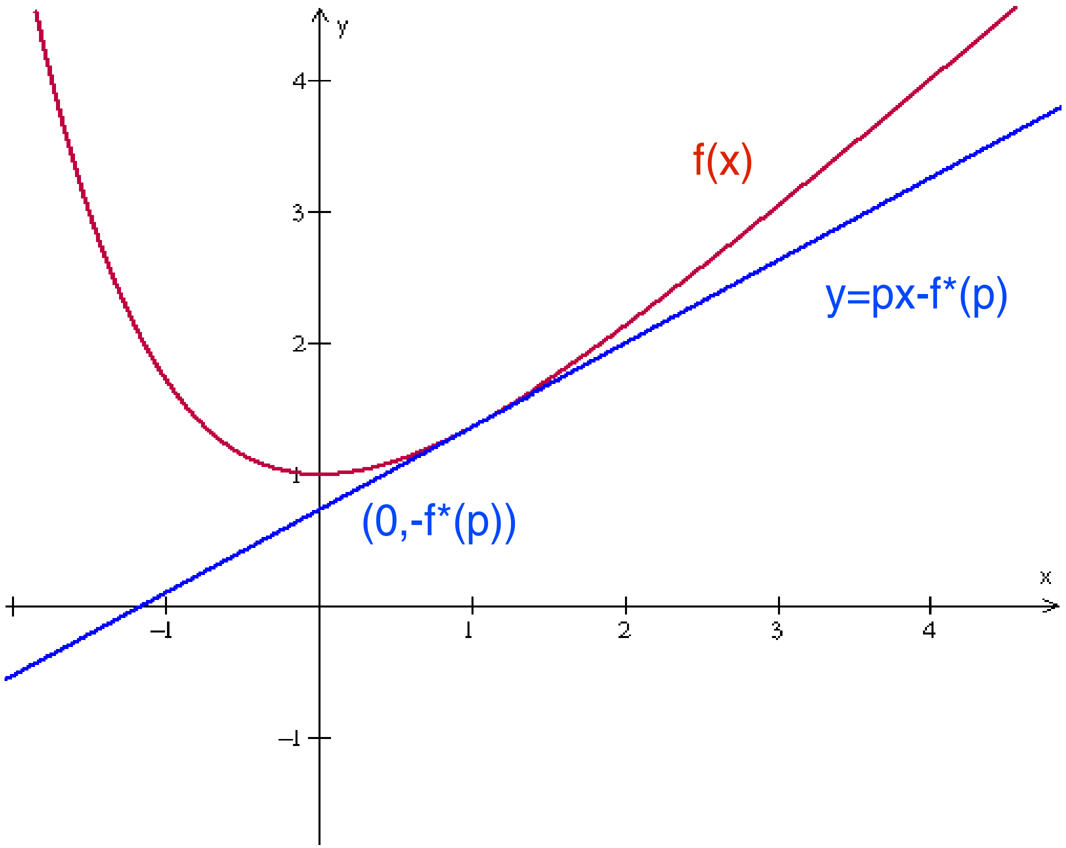
$d_\mu : h \geq 0 \mapsto \dim_{\mathcal{H}} E_\mu(h)$, where $E_\mu(h) := \{x \in [0,1]^d : h_\mu(x) = h\}$.

The $L^q$-spectrum for a probability measure $\mu \in \mathcal{M}([0,1]^d)$.

Recall $\mathbb{Z}_j = \{0,1,\cdots,2^j-1\}^d$. Let $\mathcal{G}_j$ be the partition of $[0,1)^d$ into dyadic boxes: $\mathcal{G}_j$ is the set of all cubes $I_{j,\mathbf{k}} \overset{\text{def}}{=} \prod_{i=1}^{d} [k_i 2^{-j}, (k_i+1)2^{-j})$,

where $\mathbf{k} := (k_1, k_2, \cdots, k_d) \in \mathbb{Z}_j$.

The $L^q$-spectrum of a measure $\mu \in \mathcal{M}([0,1]^d)$ is the mapping defined for any $q \in \mathbb{R}$ by $\tau_\mu(q) = \liminf_{j\to\infty} -\frac{1}{j} \log_2 s_j(q)$ where $s_j(q) = \sum_{Q\in\mathcal{G}_j,\ \mu(Q)\neq 0} \mu(Q)^q$.

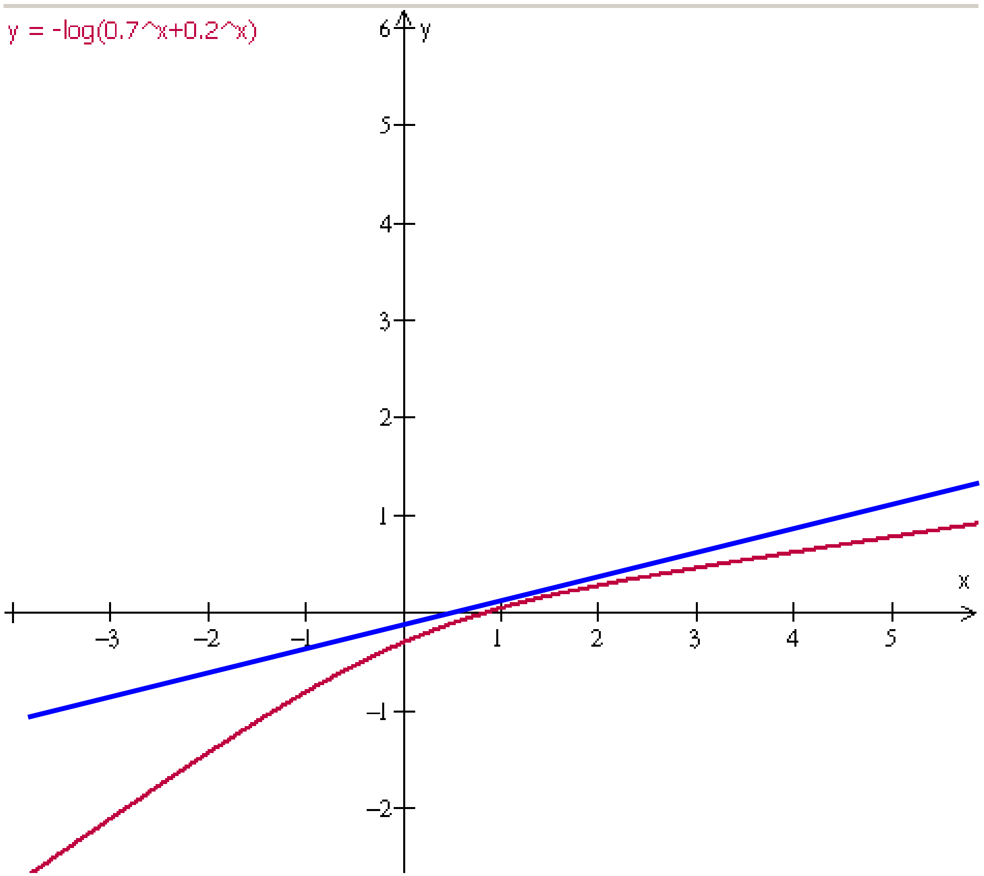



## The Legendre transform

Suppose $f$ is convex (concave up) on an interval.

Its Legendre transform $f^*(p) = \sup_{x \in \mathbb{R}} px - f(x)$.

In the strictly convex, differentiable case we also have $f^{**}(x) = x$.




If $f$ is concave (down) on an interval then a suitable variant is used.

$f^*(p) = \inf_{x \in \mathbb{R}} px - f(x)$.

The $L^q$-spectrum of a measure $\mu \in \mathcal{M}([0,1]^d)$ is the mapping defined for any $q \in \mathbb{R}$ by

$$\tau_\mu(q) = \liminf_{j \to \infty} -\frac{1}{j} \log_2 s_j(q)$$

where $s_j(q) = \sum_{Q \in \mathcal{G}_j,\ \mu(Q) \neq 0} \mu(Q)^q.$

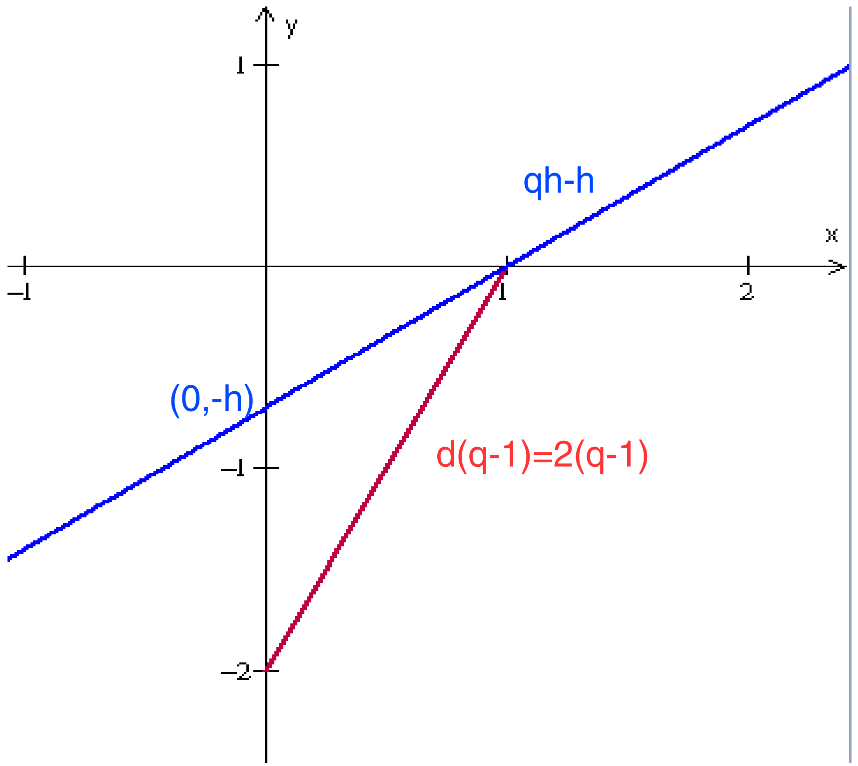
Recall:

$$\tau_\mu(q) = \liminf_{j\to\infty} -\frac{1}{j}\log_2 s_j(q) \quad \text{where} \quad s_j(q) = \sum_{Q\in\mathcal{G}_j,\ \mu(Q)\neq 0} \mu(Q)^q.$$

It is well-known that the Legendre transform of $\tau_\mu$ serves as upper bound for the multifractal spectrum $d_\mu$:

For every $h \geq 0$, $d_\mu(h) \leq (\tau_\mu)^*(h) := \inf_{q\in\mathbb{R}}(qh - \tau_\mu(q))$.

One would like to to prove that for specific measures (like self-similar measures etc.) the upper bound above turns out to be an equality. When there is an equality at exponent $h \geq 0$, the measure is said to *satisfy the multifractal formalism* at $h$.



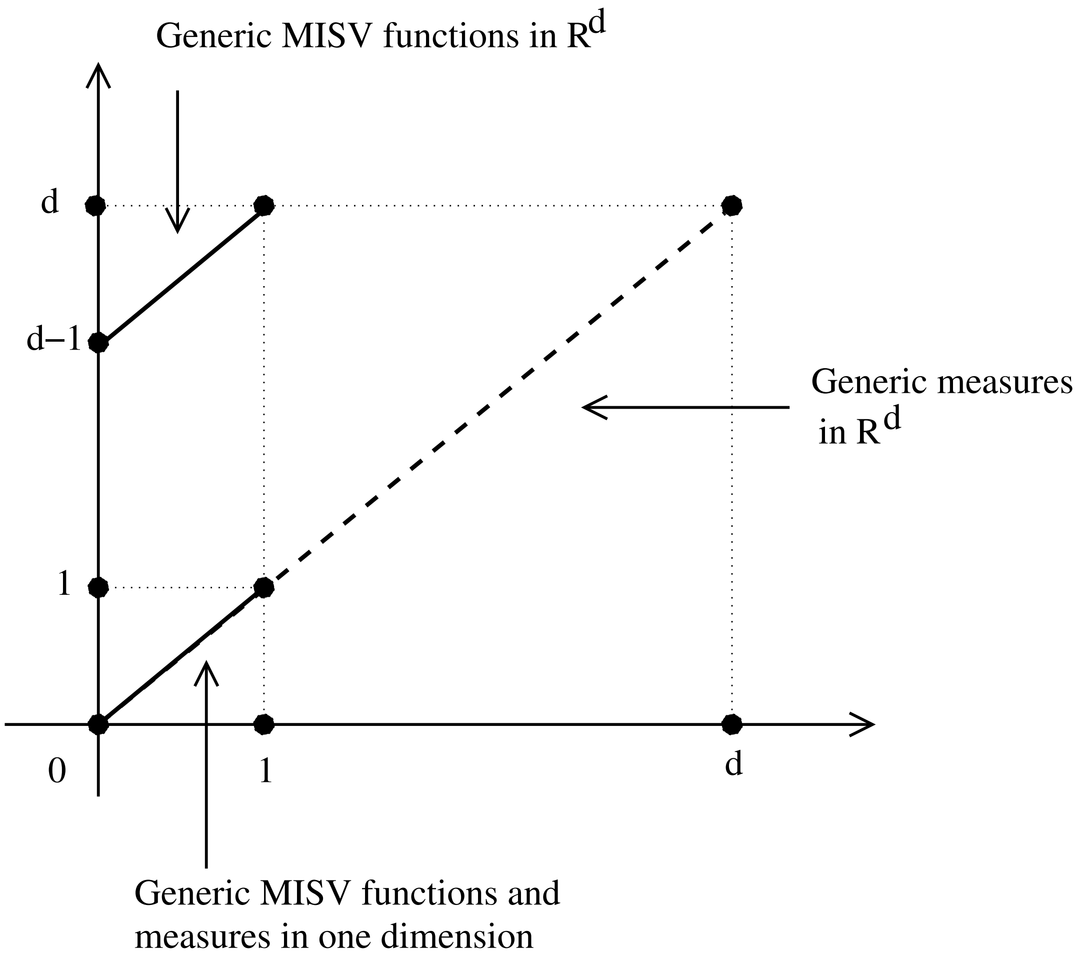

**T**.: *There is a dense $G_\delta$ set $\mathcal{R}$ included in $\mathcal{M}([0,1]^d)$ such that for every measure $\mu \in \mathcal{R}$, we have $\forall\ h \in [0,d], \quad d_\mu(h) = h$, and $E_\mu(h) = \emptyset$ if $h > d$.*

*In particular, for every $q \in [0,1]$, $\tau_\mu(q) = d(q-1)$, and $\mu$ satisfies the multifractal formalism at every $h \in [0,d]$, i.e. $d_\mu(h) = \tau_\mu^*(h)$.*

Recall $(\tau_\mu)^*(h) := \inf_{q\in\mathbb{R}}(qh - \tau_\mu(q))$

**T**.: *There is a dense $G_\delta$ set $\mathcal{R}$ included in $\mathcal{M}([0,1]^d)$ such that for every measure $\mu \in \mathcal{R}$, we have* $\forall\ h \in [0,d], \quad d_\mu(h) = h,$ *and $E_\mu(h) = \emptyset$ if $h > d$.*
*In particular, for every $q \in [0,1]$, $\tau_\mu(q) = d(q-1)$, and $\mu$ satisfies the multifractal formalism at every $h \in [0,d]$, i.e. $d_\mu(h) = \tau_\mu^*(h)$.*
We note that sometimes there is a slight difference in notation in since there is a negative sign in the definition of $\tau_\mu(q)$.

We conjecture that similar properties hold on all compact sets of $\mathbb{R}^d$.
**Conjecture:** *For any compact set $K \subset \mathbb{R}^d$, there exists a constant $0 \leq D \leq d$ such that typical measures $\mu$ (in the Baire sense) in $\mathcal{M}(K)$ satisfy: for every $h \in [0,D]$, $d_\mu(h) = h$, and if $h > D$, $E_\mu(h) = \emptyset$.*
Whether $D$ should be the Hausdorff dimension of $K$ or the lower boxdimension of $K$ (or another dimension) is not obvious for us at this point.




Compare previous results with the results for MISV functions:

If $\mu$ is supported in $[0,1]^d$, then the function $f : [0,1]^d \to \mathbb{R}$ defined as $f(x_1, x_2, ..., x_d) = \mu([0,x_1] \times [0,x_2] \times ... \times [0,x_d])$ is increasing in several variables.

For every measure, one has the upper bound

$d_\mu(h) \leq \min(h,d)$ for every $h \geq 0$,

and for typical measures supported in $[0,1]^d$,

$d_\mu(h) = h$ for every $0 \leq h \leq d$, and $E_\mu^h = \emptyset$ if $h > d$.

This is in sharp contrast with typical continuous MISV functions when $d \geq 2$, whose local behavior is "worse" (the level sets of the Hölder exponents smaller than 1 may have a bigger value).

## Dimensions of sets and measures

In $\mathbb{R}^d$ we will use the metric coming from the supremum norm, that is, for $x, y \in \mathbb{R}^d$, $\rho(x,y) = \max\{|x_i - y_i| : i = 1, ..., d\}$.
The Hausdorff measures are denoted by $\mathcal{H}^s(E)$ and Hausdorff dimensions by $\dim_{\mathcal{H}}(E)$ of a set $E$.
For a Borel measure $\mu \in \mathcal{M}([0,1]^d)$, one defines the dimension of $\mu$ as
$\dim_{\mathcal{H}}(\mu) := \sup\{s : h_\mu(x) \geq s \text{ for } \mu\text{-a.e. } x\}$.
It is known that $\dim_{\mathcal{H}}(\mu) = \inf\{\dim_{\mathcal{H}}(E) : E \subset [0,1] \text{ Borel and } \mu(E) > 0\}$.
The following property will be particularly relevant:
if $\dim_{\mathcal{H}}(\mu) \geq h$, then for every Borel set $E \subset [0,1]$
of dimension strictly less than $h$, $\mu(E) = 0$.

**Proposition.**: *Let $\mu \in \mathcal{M}([0,1]^d)$ and*
$\widetilde{E_\mu}(h) = \{x \in [0,1]^d : h_\mu(x) \leq h\} \supset E_\mu(h)$.
*For every $h \geq 0$, $d_\mu(h) = \dim_{\mathcal{H}} E_\mu(h) \leq \dim_{\mathcal{H}} \widetilde{E_\mu}(h) \leq \min(h,d)$.*
This follows for instance from the well-known result that for
$\widetilde{E_\mu}(h) = \{x \in [0,1]^d : h_\mu(x) = \underline{\dim}_{\text{loc}}\mu(x) \leq h\}$ we have $\dim_{\mathcal{H}} \widetilde{E_\mu} \leq h$.

From

**Proposition**.: *Let* $\mu \in \mathcal{M}([0,1]^d)$ *and*

$\widetilde{E_\mu}(h) = \{x \in [0,1]^d : h_\mu(x) \leq h\} \supset E_\mu(h).$

*For every* $h \geq 0$, $d_\mu(h) = \dim_{\mathcal{H}} E_\mu(h) \leq \dim_{\mathcal{H}} \widetilde{E_\mu}(h) \leq \min(h,d)$.

we deduce that in the main theorem for typical measures, $\tau_\mu(q) = d(q-1)$ for all $q \in [0,1]$:

**Corollary.** *Assume that, (as stated in the first half of the main th.)*

$(\star)\ \forall\ h \in [0,d], \qquad d_\mu(h) = h,$

*holds true for a probability measure* $\mu$ *on* $[0,1]^d$*.*

*Then* $\tau_\mu(q) = d(q-1)$ *for all* $q \in [0,1]$*.*

**Proof**.: Since $\mathcal{G}_j$ has $2^{dj}$ many cubes in $[0,1]^d$ by using Hölder's inequality for $0 < q < 1$

$$s_j(q) \stackrel{\text{def}}{=} \sum_{Q \in \mathcal{G}_j,\ \mu(Q) \neq 0} \mu(Q)^q \leq \Big(\sum_{Q \in \mathcal{G}_j} \mu(Q)^{q/q}\Big)^q \Big(\sum_{Q \in \mathcal{G}_j} 1^{1/(1-q)}\Big)^{1-q} = 1 \cdot (2^{jd})^{1-q}.$$

This implies $\tau_\mu(q) \geq d(q-1)$.

One could also notice that $\tau_\mu(0) = -d$, $\tau_\mu(1) = 0$ and

$\tau_\mu \stackrel{\text{def}}{=} \liminf_{j\to\infty} -\frac{1}{j} \log_2 s_j(q)$ is a concave map on the interior of its support and hence $\tau_\mu(q) \geq d(q-1)$ for all $q \in [0,1]$.

**Corollary.** *Assume that, (as stated in the first half of the main th.)*

$(\star)\ \forall\ h \in [0,d], \qquad d_\mu(h) = h,$

*holds true for a probability measure* $\mu$ *on* $[0,1]^d$.

*Then* $\tau_\mu(q) = d(q-1)$ *for all* $q \in [0,1]$.

We have already proved $\tau_\mu(q) \geq d(q-1)$ for all $q \in [0,1]$.
Assume now that $(\star)$ holds true for $\mu$.
Proceeding towards a contradiction suppose that
there exists $q' \in (0,1)$ such that $\tau_\mu(q') > d(q'-1)$.
By concavity of $\tau_\mu(q)$ there exists $d' < d$
such that $\tau_\mu(q) > d'(q-1)$ for all $q \in (q',1)$.
Hence for $d' < h < d$ by the well-known estimate mentioned previously
$d_\mu(h) \leq (\tau_\mu)^*(h) := \inf_{q\in\mathbb{R}} (qh - \tau_\mu(q))$ and by $(\star)$ we have
$h = d_\mu(h) \leq \inf_{q\in\mathbb{R}} (qh - \tau_\mu(q)) \leq \inf_{q\in(q',1)} (qh - d'(q-1)) = \inf_{q\in(q',1)} (q(h-d') + d') < h$, a contradiction. This concludes the proof.