Multivariate Davenport series

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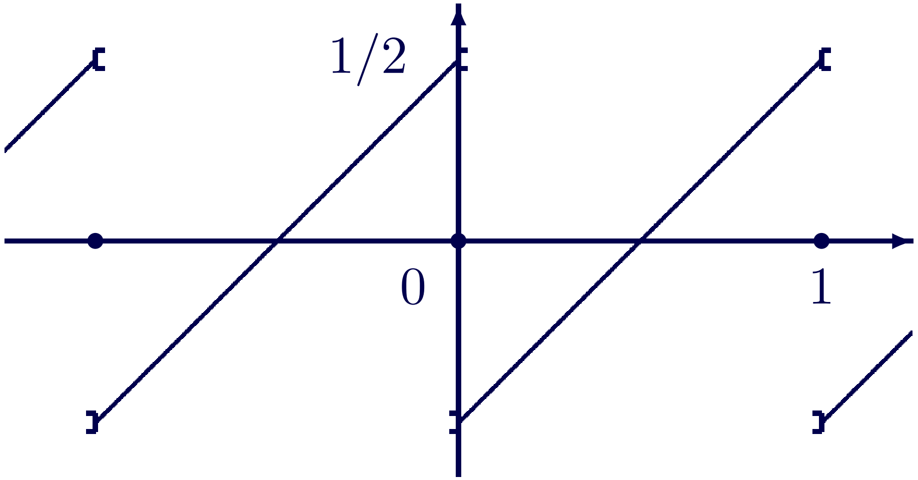
We investigate the regularity properties of the series

$$f(x) = \sum_{q \in \mathbb{Z}^d} a_q \{q \cdot x\}, \qquad x \in \mathbb{R}^d, \quad (a_q)_{q \in \mathbb{Z}^d} \in \ell^1$$

where $q \cdot x$ is the usual inner product in $\mathbb{R}^d$ and

$$\{y\} = \begin{cases} y - \lfloor y \rfloor - 1/2 & \text{if } y \notin \mathbb{Z} \\ 0 & \text{else} \end{cases}$$

- odd and one-periodic;
- jumps of magnitude one at every integer



In the one-dimensional setting, particular cases have been considered by Riemann in his "Habilitationsschrift" (1854) and by Hecke (1921): $\sum_{n=1}^{\infty} \frac{\{nx\}}{n^s}$. The general case was first considered by H. Davenport (1936), and then by R. de la Bretèche and G. Tenenbaum (2004), and by S. Jaffard (2004).

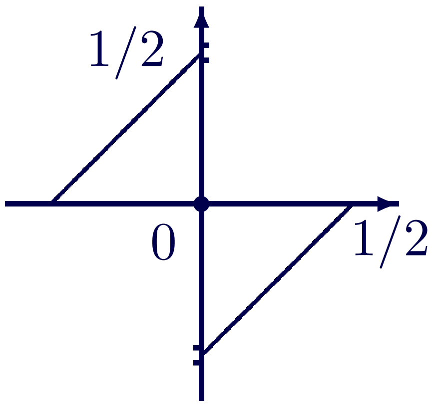
Discontinuities of Davenport series

As $\{-y\} = -\{y\}$, the decomposition

$$f(x) = \sum_{q \in \mathbb{Z}^d} a_q \{q \cdot x\} \tag{1}$$

is not unique. To ensure uniqueness, we assume that $(a_q)_{q \in \mathbb{Z}^d}$ is an odd sequence (replacing a_q by $(a_q - a_{-q})/2$ does not change (1)). We may write

$$\begin{aligned} f(x) &= \sum_{\substack{p \in \mathbb{Z} \\ q \in \mathbb{Z}^d \setminus \{0\}}} a_q \varphi(q \cdot x - p) \\ &= \sum_{(p,q) \in \mathbb{P}_d} \sum_{k=1}^{\infty} a_{kq} \varphi(k(q \cdot x - p)), \end{aligned}$$



where $(p, q) \in \mathbb{P}_d$ means $\gcd(p, q_1, \ldots, q_d) = 1$, and where $\varphi(y) = \{y\} \mathbb{1}_{\{|y| < 1/2\}}$. Hence, f has a discontinuity of magnitude

$$A_q = 2 \left| \sum_{k=1}^{\infty} a_{kq} \right| \quad \text{on each hyperplane} \quad H_{q,p} = \{x \in \mathbb{R}^d \mid p = q \cdot x\}.$$

Comparison with Lévy random fields

- They extend the notion of Lévy processes to the multivariate setting (T. Mori, 1992); in particular, the trace of a Lévy field along any fixed half-line is a Lévy process.
- They admit a remarkable Lévy-Itō decomposition as a sum of a linear drift, a Gaussian component and a jump component. The jump component is

$$L_\nu(x) = \sum_{j=1}^{\infty} \Bigg(\underbrace{\sum_{|Y_n| \in \mathcal{I}_j} Y_n \, \mathbb{1}_{\{\mathrm{P}_n < S_n \cdot x\}}}_{\text{piecewise constant}} - \underbrace{\int_{\substack{s \in \mathbb{S}^{d-1} \\ y \in \mathcal{I}_j}} y(s \cdot x)\, \nu(\mathrm{d}s, \mathrm{d}y)}_{\text{linear compensator}} \Bigg).$$

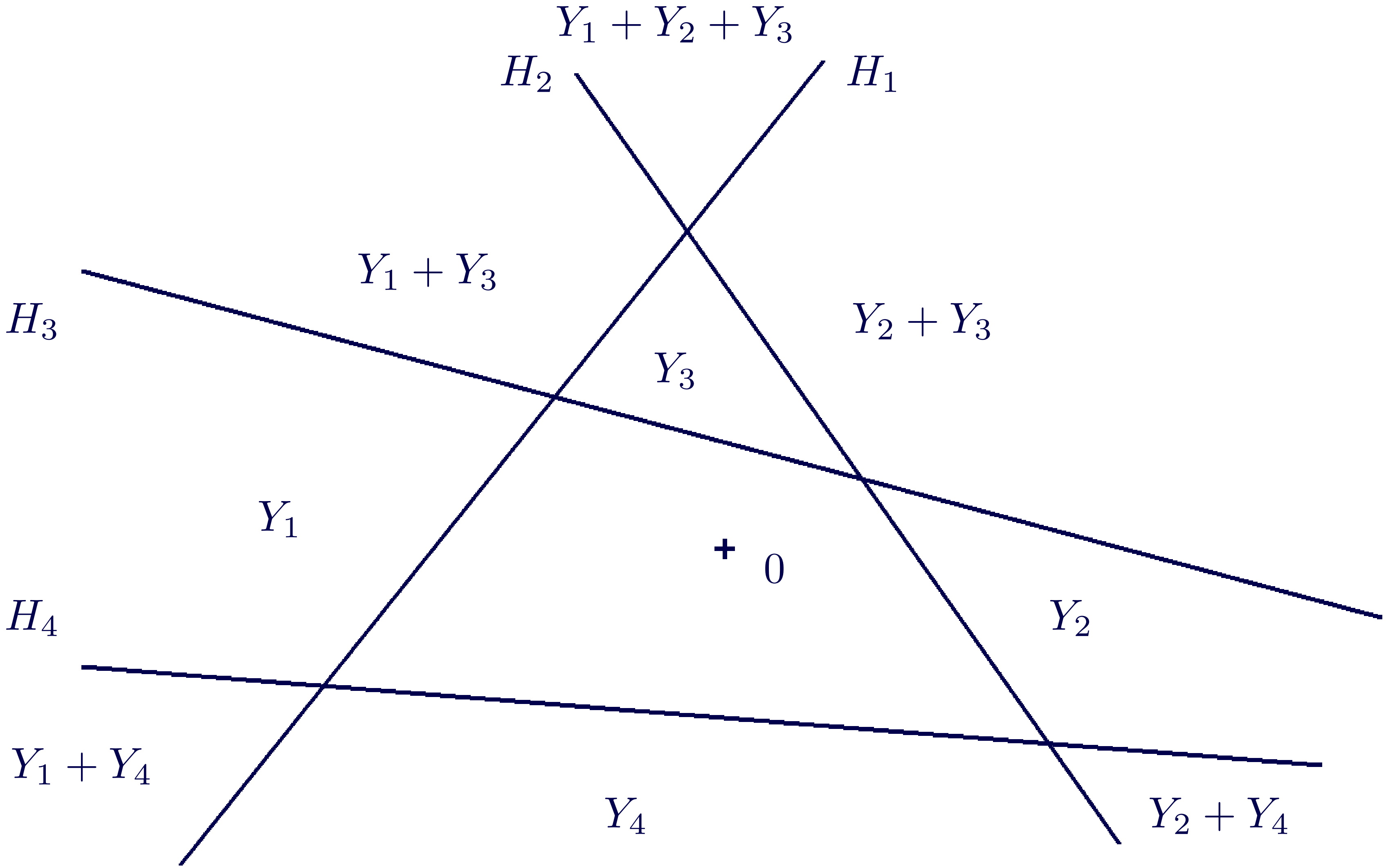
- $\mathcal{I}_j = (2^{-j}, 2^{-j+1}]$ and (P_n, S_n, Y_n) are the atoms of a Poisson random measure with intensity $\mathcal{L} \otimes \nu$, where ν is a nonnegative Borel measure on $\mathbb{S}^{d-1} \times [-1, 1]$ which is symmetric (i.e. invariant under $(s, y) \mapsto (-s, -y)$) and such that

$$\int_{\substack{s \in \mathbb{S}^{d-1} \\ |y| \le 1}} y^2 \, \nu(\mathrm{d}s, \mathrm{d}y) < \infty;$$

  ν is the analog of the Lévy measure: $\nu(\mathrm{d}s, \mathrm{d}y)$ describes the amount of hyperplanes orthogonal to s where a jump of size y occurs.

Comparison with Lévy random fields

$\sum_{|Y_n| \in \mathcal{I}_j} Y_n \mathbb{1}_{\{\mathrm{P}_n < S_n \cdot x\}}$ has a discontinuity of size $|Y_n|$ on $H_n = \{x \in \mathbb{R}^d \mid P_n = S_n \cdot x\}$.



Multifractal analysis of functions

Let $f : \mathbb{R}^d \to \mathbb{R}$ be locally bounded.

- The *Hölder exponent* $\alpha_f(x_0)$ is the supremum of all $\alpha \geq 0$ such that

$$\exists P_{x_0} \text{ polynomial} \quad \exists C \quad \forall x \text{ near } x_0 \qquad |f(x) - P_{x_0}(x)| \leq C\|x - x_0\|^\alpha.$$

- The *iso-Hölder sets* of f are

$$E_f(h) = \{x \in \mathbb{R}^d \mid \alpha_f(x) = h\}.$$

  and its *local spectrum of singularities* is $d_f(h, W) = \dim_H(E_f(h) \cap W)$.

- The *singularity sets* of f are

$$E'_f(h) = \{x \in \mathbb{R}^d \mid f \text{ is continuous at } x \text{ and } \alpha_f(x) \leq h\}.$$

Multifractal analysis of Lévy fields

The jump component is a *homogeneous multifractal process*.

Theorem (D.,Jaffard, 2011)**.** *Almost surely, for any nonempty open* $W \subseteq \mathbb{R}^d$*,*

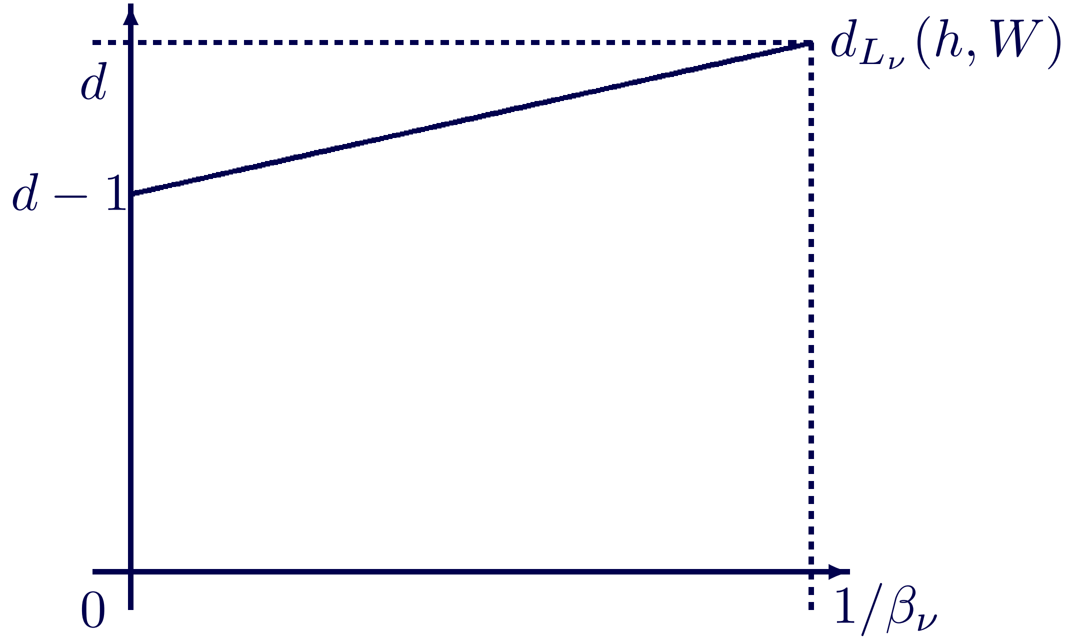
$$d_{L_\nu}(h, W) = d - 1 + \beta_\nu h, \qquad 0 \le h \le 1/\beta_\nu,$$

and moreover $E_{L_\nu}(h) = \emptyset$ *for* $h > 1/\beta_\nu$*.*

This holds in the generic case where $0 < \beta_\nu < 2$. Here, β_ν is the *index* of ν:

$$\beta_\nu = \inf\left\{\gamma > 0 \middle| \int_{\substack{s\in\mathbb{S}^{d-1}\\ y\in(0,1]}} y^\gamma \nu(\mathrm{d}s, \mathrm{d}y) < \infty\right\}.$$

(Analog of the Blumenthal-Getoor index in the multivariate case)



Theorem (D.,Jaffard, 2011). *Almost surely, for* $0 \leq h \leq 1/\beta_\nu$*,*

$$E'_{L_\nu}(h) \in \mathcal{G}^{d-1+\beta_\nu h}(\mathbb{R}^d),$$

where $\mathcal{G}^{d-1+\beta_\nu h}(\mathbb{R}^d)$ *is the class of sets with large intersection of K. Falconer.*

Theorem (K. Falconer, 1994). *For any* $s \in (0, d]$*, the class* $\mathcal{G}^s(\mathbb{R}^d)$ *is the maximal class of* G_δ*-subsets of* $\mathbb{R}^d$ *of Hausdorff dimension at least* s *that is closed under countable intersections and similarities.*

The above results come from the fact that the sets $E_{L_\nu}(h)$ and $E'_{L_\nu}(h)$ may be expressed in terms of

$$K_\nu(\alpha) = \left\{ x \in \mathbb{R}^d \,\middle|\, \mathrm{dist}(x, H_n) < |Y_n|^{1/\alpha} \text{ for infinitely many } n \geq 1 \right\}$$

and from a precise understanding of the size and large intersection properties of the latter sets. (Recall that there is a jump of size Y_n on the hyperplane H_n.)

Connection with an approximation problem

Proposition. *A.s., for* $0 \le h \le 1/\beta_\nu$*,*

$$E'_{L_\nu}(h) = (\mathbb{R}^d \setminus J_\nu) \cap \bigcap_{\alpha > h} \downarrow K_\nu(\alpha),$$

with

$$K_\nu(\alpha) = \left\{ x \in \mathbb{R}^d \;\middle|\; \operatorname{dist}(x, H_n) < |Y_n|^{1/\alpha} \text{ i.o.} \right\}$$

and $J_\nu = \bigcup_{n \ge 1} H_n$ *(jump locations of* L_ν*).*

Remark. *Sets comparable to* $K_\nu(\alpha)$ *appear in: B. Mandelbrot, The Fractal Geometry of Nature, 1982*

- ☐ Do the multivariate Davenport series satisfy the same properties?

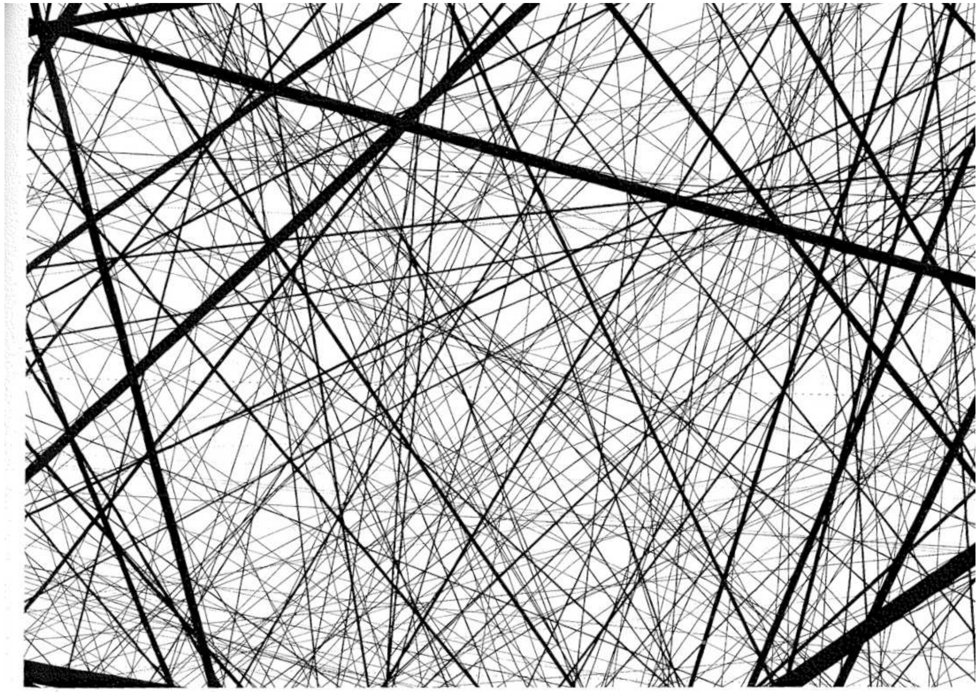


Plate 285 ¤ RANDOM PATTERN OF STREETS

As noted in Chapter 8, it is regrettable that the Cantor dust should be so hard to illustrate directly. However, it can be visualized indirectly as the intersection of the triadic Koch curve with its base. And in the same way the Lévy dust can be imagined indirectly. On this plate, the black street-like stripes are placed at random, and in particular their directions are isotropic. Their widths follow a hyperbolic distribution and rapidly become so thin that they cannot be drawn. Asymptotically, the white remainder set (the "blocks of houses") is of zero area and of dimension D less than 2.

As long as the remaining blocks of houses have a dimension D>1, their intersection by an arbitrary line is a Lévy dust of dimension D−1. On the other hand, if D<1, the intersection is almost certainly empty. This result is, however, not very apparent here because the construction could not be carried far enough.

Chapter 33 provides a better illustration. When the tremas subtracted from the plane are random discs as exemplified by Plates 306 to 309, the trema fractals' intersections with straight lines are Lévy dusts. ■

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Lemma. *Let $f : \mathbb{R}^d \to \mathbb{R}$ be locally bounded and let $x_0 \in \mathbb{R}^d$. Then,*

$$\alpha_f(x_0) \leq \liminf_{s \to x_0} \frac{\log \Delta_f(s)}{\log |s - x_0|} \qquad \textit{with} \qquad \Delta_f(s) = \limsup_{x \to s} f(x) - \liminf_{x \to s} f(x).$$

Recall that the Davenport series $f(x) = \sum_{q \in \mathbb{Z}^d} a_q \{q \cdot x\}$ has a discontinuity of magnitude

$$A_q = 2 \left| \sum_{k=1}^{\infty} a_{kq} \right| \quad \text{on each hyperplane} \quad H_{q,p} = \{x \in \mathbb{R}^d \mid p = q \cdot x\},$$

for $p \in \mathbb{Z}$ and $q \in \mathbb{Z}^d \setminus \{0\}$ with $\gcd(p, q_1, \ldots, q_d) = 1$. Therefore, for all $x_0 \in \mathbb{R}^d$,

$$\alpha_f(x_0) \leq \liminf_{q : A_q > 0} \frac{\log A_q}{\log d_q^{\mathbb{P}}(x_0)} \qquad \text{with} \qquad d_q^{\mathbb{P}}(x_0) = \operatorname{dist}\left(x_0, \bigcup_{\substack{p \in \mathbb{Z} \\ \gcd(p,q)=1}} H_{q,p} \right).$$

Connection with Diophantine approximation

The points at which f is not continuous are located on the set $J_f = \bigcup_{\substack{(p,q)\in\mathbb{P}_d \\ A_q>0}} H_{q,p}$. It follows from the previous lemma that

$$E'_f(h) \supseteq (\mathbb{R}^d \setminus J_f) \cap \bigcap_{\alpha>h} \downarrow K_f^{\mathbb{P}}(\alpha),$$

where

$$\begin{aligned} K_f^{\mathbb{P}}(\alpha) &= \{x \in \mathbb{R}^d \mid \operatorname{dist}(x, H_{q,p}) < A_q^{1/\alpha} \text{ for infinitely many } (p,q) \in \mathbb{P}_d\} \\ &= \{x \in \mathbb{R}^d \mid |q \cdot x - p| < |q| A_q^{1/\alpha} \text{ for infinitely many } (p,q) \in \mathbb{P}_d\}. \end{aligned}$$

- □ The precise knowledge of the size and large intersection properties of $K_f^{\mathbb{P}}(\alpha)$ would follow from the settlement of the dual Duffin-Schaeffer conjecture (1941).
- □ Even for simple examples, the lemma may not yield the correct Hölder exponent: for $\varepsilon > 0$ small enough, the function $\sum_{n\geq 1} \frac{\{nx\}}{n^{2-\varepsilon}} - \zeta(2-\varepsilon)\{x\}$ has exponent $1-\varepsilon$ at zero whereas the lemma yields the bound $\liminf_n \frac{\log(1/n^{2-\varepsilon})}{\log(1/n)} = 2-\varepsilon$.

Definition. *A multivariate Davenport series is lacunary if it may be written under the form*

$$f(x) = \sum_{n\geq 1} a_{\lambda_n}\{\lambda_n \cdot x\},$$

where $(\lambda_n)_{n\geq 1}$ *is a sequence valued in* $\mathbb{Z}^d$ *satisfying*

$$\sup_{j\geq 0} \#\{n \geq 1 \mid 2^j \leq |\lambda_n| < 2^{j+1}\} < \infty.$$

We shall assume that there is no excessive cancellation in the sums $A_q = 2\left|\sum_{q|\lambda_n} a_{\lambda_n}\right|$ in the sense that

$$(H) \qquad \limsup_q \frac{\log A_q}{\log \sup_{q|\lambda_n} |a_{\lambda_n}|} \leq 1 \qquad \left(A_q \gtrsim \sup_{q|\lambda_n} |a_{\lambda_n}|\right).$$

Connection with Diophantine approximation

Theorem (D., Jaffard, 2011)**.** *Let f be a lacunary multivariate Davenport series satisfying (H). Then, for all $x_0 \in \mathbb{R}^d$,*

$$\alpha_f(x_0) = \liminf_{q:A_q>0} \frac{\log A_q}{\log d_q^{\mathbb{P}}(x_0)} = \liminf_n \frac{\log |a_{\lambda_n}|}{\log d_{\lambda_n}(x_0)} \quad \textit{with} \quad d_{\lambda_n}(x_0) = \operatorname{dist}\left(x_0, \bigcup_{p\in\mathbb{Z}} H_{\lambda_n,p}\right).$$

It follows from the previous theorem that

$$E'_f(h) = (\mathbb{R}^d \setminus J_f) \cap \bigcap_{\alpha>h} \downarrow K_f(\alpha) \qquad \text{and} \qquad E_f(h) \setminus J_f = E'_f(h) \setminus \bigcup_{\alpha<h} \uparrow K_f(\alpha),$$

where

$$\begin{aligned} K_f(\alpha) &= \{x \in \mathbb{R}^d \mid \operatorname{dist}(x, H_{\lambda_n,p}) < |a_{\lambda_n}|^{1/\alpha} \text{ for infinitely many } (p,n) \in \mathbb{Z}\times\mathbb{N}\} \\ &= \{x \in \mathbb{R}^d \mid |\lambda_n \cdot x - p| < |\lambda_n||a_{\lambda_n}|^{1/\alpha} \text{ for infinitely many } (p,n) \in \mathbb{Z}\times\mathbb{N}\}. \end{aligned}$$

Multifractal analysis of Davenport series

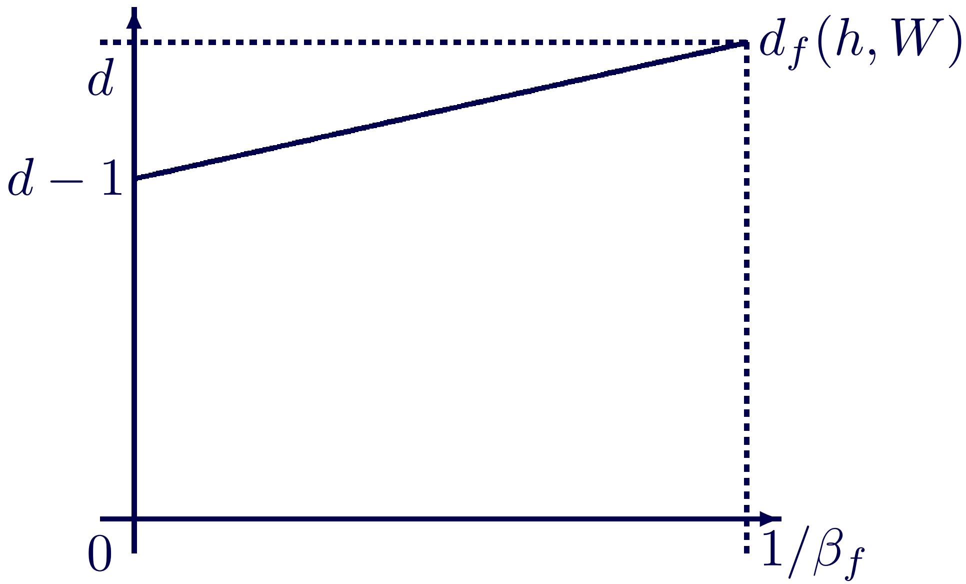
Theorem (D., Jaffard, 2011). *Let f be a lacunary multivariate Davenport series satisfying (H) and let*

$$\beta_f = \limsup_n \frac{\log(1/|\lambda_n|)}{\log|a_{\lambda_n}|}.$$

If $0 < \beta_f < \infty$, then for any nonempty open set $W \subseteq \mathbb{R}^d$,

$$d_f(h, W) = d - 1 + \beta_f h, \qquad 0 \leq h \leq 1/\beta_f,$$

and moreover $E_f(h) = \emptyset$ for $h > 1/\beta_f$.



Theorem (D., Jaffard, 2011). *Under the previous assumptions, for $0 \leq h \leq 1/\beta_f$,*

$$E'_f(h) \in \mathcal{G}^{d-1+\beta_f h}(\mathbb{R}^d),$$

where $\mathcal{G}^{d-1+\beta_f h}(\mathbb{R}^d)$ is the class of sets with large intersection of K. Falconer.

References

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