Statistical estimation of the intermittency coefficient of a random cascade

Laurent DUVERNET (Université Paris 7)

Fractals and Related Fields II
Porquerolles
June 13-17, 2011

The setting

Estimation procedures and convergence rates

The setting

Estimation procedures and convergence rates

Statistical inference for multifractal processes

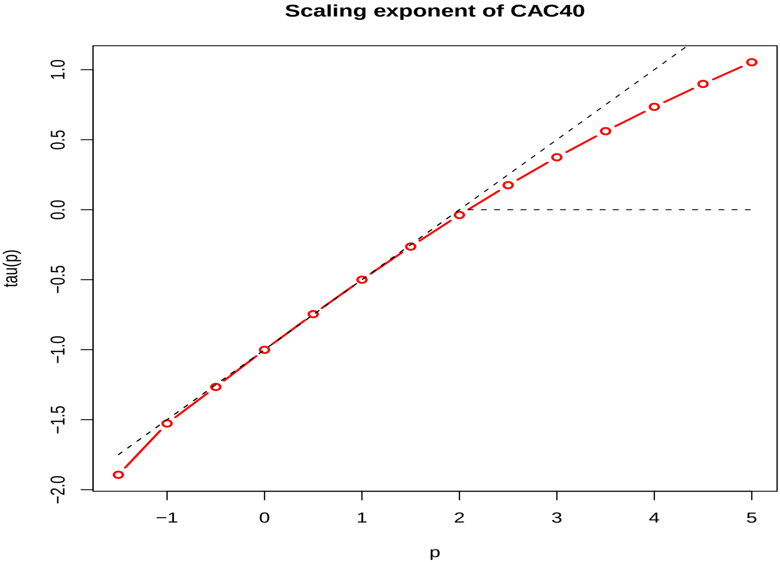
- Let $X = \big(X(t),\ t \geq 0\big)$ a real-valued random process s.t.

$$\mathbb{E}\big[|X(t+s) - X(t)|^p\big] \approx s^{\zeta(p)} \quad \text{as } s \to 0.$$

- Hölder inequality: $p \mapsto \zeta(p)$ is a **concave function.** We say that X is **monofractal** if ζ is **linear**, and X is **multifractal** if ζ is **strictly concave.**
  $\to$ multifractal formalism, Hölder regularity of the sample paths of X.
- Suppose that we have some data $X_0,\ X_{1/n},\ \ldots,\ X_1$.
  Can we recognize if X is mono- or multifractal? and how accurately can we reconstruct ζ as the number of data n grows ?

$\to$ **Applications to real data:** turbulence, finance...

Scaling exponent on financial data



Pointwise estimation *vs.* cumulant estimation

Estimating $\zeta(p)$ for all p in some real interval:

- First possibility: find consistent estimates of $\zeta(p_1), \ldots, \zeta(p_{u_n})$ with $u_n \to +\infty$ as the number of data n grows.
- Second possibility: estimate $\zeta(0)$, $\zeta'(0)$, $\zeta''(0)$, ...
  Intermittency coefficient: $\zeta''(0)$ which indicates whether ζ is **linear** or not.
- Here I consider processes of random cascades that satisfy a **scaling property**: for some interval I and all $r \in (0,1)$

$$\big(X(rt),\, t \in I\big) \stackrel{d}{=} r W_r \big(X(t),\, t \in I\big)$$

with W_r a positive r.v. indep. of X. Then $\zeta^{(k)}(0)$ can be recovered from $\mathbb{E}\left[\log^k W_r\right]$.

Mandelbrot cascades [Mandelbrot 1974, Kahane and Peyrière 1976]

A Mandelbrot cascade is a **continuous, nonnegative and increasing** process $\big(X(t),\, t \in [0, T]\big)$. Let W a positive r.v. with $\mathbb{E}\big[W\big] = 1$, and

$$(W_i,\, i \in \{0, 1\}^k,\, k \in \mathbb{N})$$

i.i.d. copies of W. $X(t)$ is defined as the limit of the sequence

$$X_1(t) = \int_0^t \big(W_0 1_{u\in[0,1/2]} + W_1 1_{u\in[1/2,1]}\big)\mathrm{d}u$$

$$X_2(t) = \int_0^t \big(W_0 W_{00} 1_{u\in[0,1/4]} + W_0 W_{01} 1_{u\in[1/4,1/2]} + W_1 W_{10} 1_{u\in[1/2/3/4]} + W_1 W_{11} 1_{u\in[3/4,1]}\big)\mathrm{d}u$$

$$X_3(t) = \dots$$

Log-normal cascades

- More elaborate framework: **grid free** constructions of random cascades: Kahane (1985), Barral and Mandelbrot (2002), Bacry and Muzy (2003)...
- Popular setting: **"log-normal" cascades**: log W is a Gaussian random variable with variance λ^2.
- They are multifracal processes with scaling exponent

$$\zeta(p) = p - \lambda^2 p(p-1)/2.$$

- Hence estimating the **function** ζ is the same as simply estimating the **scalar** λ^2 – the *intermittency coefficient*.

The setting

Estimation procedures and convergence rates

Estimation based on the empirical moments

Idea: approximate the true theoretical moments by empirical moments.

- Let

$$S_n(p) = \frac{1}{n}\sum_{k=0}^{n-1} |X((k+1)/n) - X(k/n)|^p.$$

Then from the multifractal property

$$S_n(p) \approx \mathbb{E}\left[|X((k+1)/n) - X(k/n)|^p\right] = n^{-1+\lambda^2 p(p-1)/2}.$$

- Hence define

$$\widetilde{\lambda}_n^{2,p} = \frac{2}{p(p-1)}\left(1 + \frac{\log S_n(p)}{\log n}\right)$$

or

$$\hat{\lambda}_n^{2,p} = \frac{2}{p(p-1)}\left(1 + \frac{\log S_{2n}(p) - \log S_n(p)}{\log 2}\right).$$

Convergence rate

Theorem 1 (Ossiander and Waymire 2000, D. 2011, Ludeña and Soulier 2011)

Let X a log-normal Mandelbrot cascade, or a log-normal multifracal random measure (Bacry and Muzy 2003). Then for some $p^* > 1$ and all $p \in (0, p^*)$, $p \neq 1$,

$$|\hat{\lambda}_n^{2,p} - \lambda^2| \asymp n^{-1/2+\lambda^2 p^2/2}.$$

- **Slower rate of convergence** than the usual parametric rate $1/\sqrt{n}$.

Are there estimation procedures that converge faster ?

Logarithms of increments

Abry, Jaffard, Roux and Wendt (2007); Bacry, Kozhemyak and Muzy (2008) $\rightarrow$ estimates based on the properties of the **log** of the increments (or wavelet coef., or wavelete leaders.)

- From the scaling property:

$$\big(X(t/n),\ t \in I\big) \stackrel{d}{=} 1/n\, W_{1/n} \big(X(t),\ t \in I\big)$$

  with $\log W_{1/n}$ a **Gaussian r.v.** with variance $\lambda^2 \log n$.

- Let $x_{n,k} = \log|X((k+1)/n) - X(k/n)|$, then

$$x_{n,k} \stackrel{d}{=} -\log(n) + w_{n,k} + m_{n,k},\ 0 \leq k \leq n-1,$$

  with $(w_{n,k})_k$ a **Gaussian sequence** with covariance $\mathbb{C}\mathrm{ov}[w_{n,k}, w_{n,k'}] \approx c + \lambda^2 \log(n/(|k-k'|+1))$, while $(m_{n,k})$ can be considered as a "noise".

Using the moments of the log of the increments for estimating λ^2

From the previous decomposition, we hope to recover λ^2 from the second order properties of $w_{n,k}$.

- Abry *et al.* 2007: estimator based on the **empirical variance** of the $x_{n,k}$'s.
  $\rightarrow$ I find a rate of convergence $\frac{\log n}{\sqrt{n}}$.
- Bacry *et al.* 2008: estimator based on the **empirical covariance** of the $x_{N,k}$'s.
  $\rightarrow$ rate of convergence $\frac{\sqrt{\log n}}{\sqrt{n}}$.
- D., 2011: estimator based on the **empirical variance** of $x_{N,k+1} - x_{N,k}$.
  $\rightarrow$ rate of convergence $\frac{1}{\sqrt{n}}$.

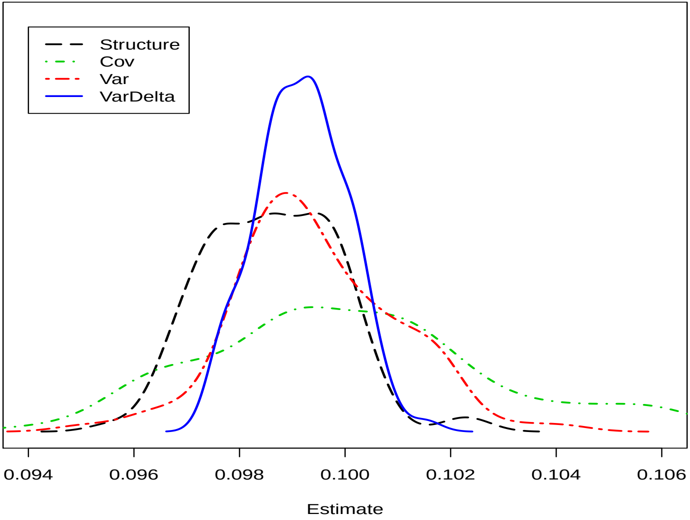


Figure: Empirical distribution of the estimators for 100 simulations of a log-normal MRM; $\lambda^2 = 0.1$; $N = 32\,768$.

Conclusion

- How accurately can we characterize the multifractality of some data when the number of data grows?
- In the "simple" case of log-normal random cascades, multifractality is characterized by a single real number, the intermittency coefficient.
- We find different convergence rates for different estimation procedures of this coefficient.
- In particular, the "usual" approach based on the empirical moments of the increments of the process is sub-optimal.