Multifractal analysis of Bernoulli convolutions associated with Salem numbers

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Fractals and Related Fields II,
Porquerolles - France, June 13th-17th 2011

Bernoulli convolutions

- For any $\lambda \in (0,1)$, the Bernoulli convolution μ_λ is the distribution of $\sum_{n=0}^{\infty} \epsilon_n \lambda^n$, where the coefficients ϵ_n are either -1 or 1, chosen independently with probability $\frac{1}{2}$ for each.
- $\mu_\lambda = \bigotimes_{n=0}^{\infty} \frac{1}{2}(\delta_{-\lambda^n} + \delta_{\lambda^n})$.

- μ_λ can be expressed as the self-similar measure satisfying the equation
$$\mu_\lambda = \frac{1}{2}\mu_\lambda \circ S_1^{-1} + \frac{1}{2}\mu_\lambda \circ S_2^{-1}, \qquad (1)$$
where $S_1(x) = \lambda x - 1$ and $S_2(x) = \lambda x + 1$.

- When $\lambda \in (0, 1/2)$, μ_λ is a singular measure supported on a Cantor set. When $\lambda \in [1/2, 1)$, the support of μ_λ is an interval.

An Erdös problem

- The fundamental question about μ_λ is to decide for which $\lambda \in (\frac{1}{2}, 1)$ this measure is absolutely continuous and for which λ it is singular. It is well known that for each $\lambda \in (1/2, 1)$, μ_λ is continuous, and it is either purely absolutely continuous or purely singular.

- **Solomyak (1995)** proved that μ_λ is absolutely continuous for a.e. $\lambda \in (1/2, 1)$. In the other direction, **Erdös (1939)** proved that if λ^{-1} is a **Pisot number**, i.e. an algebraic integer whose algebraic conjugates are all inside the unit disk, then μ_λ is singular.

- It is an **open problem** whether the Pisot reciprocals are the only class of λ's in $(\frac{1}{2}, 1)$ for which μ_λ is singular. This question is far from being answered.

Possible candidates for counter-examples

- There appears to be a general belief that the best candidates for counter-examples are the reciprocals of **Salem numbers**. A number $\beta > 1$ is called a Salem number if it is an algebraic integer whose algebraic conjugates all have modulus no greater than 1, with at least one of which on the unit circle.

- A well-known class of Salem numbers are the largest real roots β_n of the polynomials $x^n - x^{n-1} - \cdots - x + 1$; where $n \geq 4$.

- Indeed, when λ^{-1} is a Salem number, the Fourier transform of μ_λ has no uniform decay at infinity (**Kahane (1971)**), i.e., $\limsup_{\xi\to\infty} \widehat{\mu_\lambda}(\xi)\xi^\epsilon = \infty$ for all $\epsilon > 0$. Hence, $\frac{d\mu_\lambda}{dx} \notin C^1(\mathbb{R})$.

- Let β_n be the largest root of the polynomials $x^n - x^{n-1} - \cdots - x + 1$; where $n \geq 4$. It was shown that for any $\epsilon > 0$, the density of μ_{1/β_n}, if it exists, is not in $L^{3+\epsilon}(\mathbb{R})$ when n is large enough (**F. and Wang (2004)**).

- See **Peres-Schlag-Solomyak** (Progress in Probability, 2000), **Solomyak** (Proc. Symp. in Pure Math., 2004) for a good survey on Bernoulli convolutions.

Our target

To study the **local dimensions and the multifractal structure** of μ_λ when λ^{-1} is a **Salem number** in $(1, 2)$. Very little has been known in the literature.

Notation

Let μ be a finite Borel measure in $\mathbb{R}^d$ with compact support.

- For $x \in \mathbb{R}^d$, the **local dimension** of μ at x is defined as

$$d_\mu(x) = \lim_{r \to 0} \frac{\log \mu(B_r(x))}{\log r},$$

provided that the limit exists.

- For $\alpha \in \mathbb{R}$, the **α-level set** of μ is defined as

$$E_\mu(\alpha) = \{x \in \mathbb{R} : \ d_\mu(x) = \alpha\}.$$

- For $q \in \mathbb{R}$, the **L^q spectrum** of μ is defined as

$$\tau_\mu(q) = \liminf_{r \to 0} \frac{\log \Theta_\mu(q; r)}{\log r},$$

where $\Theta_\mu(q; r) = \sup \sum_i \mu(B_r(x_i))^q$ for $r > 0, \ q \in \mathbb{R}$, and the supremum is taken over all families of disjoint balls $\{B_r(x_i)\}_i$ with $x_i \in \mathrm{supp}(\mu)$.

Multifractal analysis

- One of the main objectives is to study the **dimension spectrum** $\dim_H E_\mu(\alpha)$ and its relation with the L^q spectrum $\tau_\mu(q)$
- A heuristic principle known as **multifractal formalism** (MF) was proposed by **Halsey et al (1986)**:

$$\dim_H E_\mu(\alpha) = \tau_\mu^*(\alpha) := \inf\{\alpha q - \tau_\mu(q) : \ q \in \mathbb{R}\}. \qquad (2)$$

- MF is valid for some good measures, including
  - **Gibbs measures for smooth conformal dynamical systems** (e.g., **Rand (1989), Pesin-Weiss (1997)**).
  - **Self-similar measures with open set condition** (**Cawley-Mauldin (1992), Brown-Michon-Peyrière(1992), Olsen (1995), Patzschke (1997)**).
  - More precisely, for these measures,
    - $\tau_\mu(q)$ is real analytic over $\mathbb{R}$;
    - $\{\alpha : \ E_\mu(\alpha) \neq \emptyset\} = [\alpha_{\min}, \alpha_{\max}]$, where

$$\alpha_{\min} = \lim_{q\to\infty} \tau_\mu(q)/q, \quad \alpha_{\max} = \lim_{q\to-\infty} \tau_\mu(q)/q.$$

    - $\dim_H E_\mu(\alpha) = \tau_\mu^*(\alpha)$ for $\alpha \in [\alpha_{\min}, \alpha_{\max}]$.

- MF is not valid for general measures. However, the upper bound $\dim_H E_\mu(\alpha) \le \tau_\mu^*(\alpha)$ always holds.
(e.g., **Lau-Ngai (1999)**).

- **Question**: Is MF valid for **Bernoulli convolutions** (self-similar measures with overlaps)?

- **Question**: Is MF valid for **Bernoulli convolutions** (self-similar measures with overlaps)?
- Difficulty:
  - hard to analyze the local behavior of μ and estimate the local dimension of μ;
  - hard to estimate the L^q-spectrum $\tau_\mu(q)$ and its regularity property.

Historic remarks: When $1/\lambda$ is a Pisot number

- Many works in the literature: e.g., Alexander-Yorke (1984), Przytycki-Urbanski (1989), Alexander-Zagier (1991), Lau (1992), Ledrappier-Porzio (1994, 1996), Strichartz-Taylor-Zhang (1995), Lau-Ngai (1998, 1999), Lalley (1998), Porzio (1998), Vershik-Sidorov (1998), F. (2003, 2005, 2009), F. & Olivier (2003), F. & Lau (2009).

- **Phase transition** for $q < 0$ in the golden ratio case ($\lambda = \frac{\sqrt{5}-1}{2}$). That is, $\tau_\mu(q)$ is not differentiable at some $q < 0$. (F., 1999, 2005).

  Similar exceptional phenomena for other self-similar measures with overlaps (e.g., Hu-Lau (2001), F. -Lau-Wang (2005), Shmerkin (2005), Testud (2006))

- So far the most complete result is the following (F., 2009):
  - $\dim_H E_\mu(\alpha) = \tau_\mu^*(\alpha)$ for $\alpha \in [\tau'_\mu(\infty), \tau'_\mu(0-)]$.
  - $\exists$ an interval $I \subset \mathrm{supp}(\mu)$ so that, for $\nu = \mu|_I$,
    - $E_\nu(\alpha) \neq \emptyset$ if and only if $\alpha \in [\tau'_\nu(+\infty), \tau'_\nu(-\infty)]$.
    - $\dim_H E_\nu(\alpha) = \tau_\nu^*(\alpha)$ for each $\alpha \in [\tau'_\nu(+\infty), \tau'_\nu(-\infty)]$.
    - $\tau_\nu(q) = \tau_\mu(q)$ for $q \geq 0$.

- The above results hold for self-similar measures with **weak separation condition** (F. & Lau (2009)); this condition was introduced by Lau-Ngai (1999).

- We point out that in the pisot case, τ_μ is **differentiable** on $(0, \infty)$ (F. (2003)); and $[\tau'_\mu(\infty), \tau'_\mu(0-)]$ contains a **neighborhood of** 1. (F. & Sidorov (2011)).

- Based on products of random matrices and the thermodynamic formalism.

Historic remarks: Non-Pisot case

- For every $\lambda \in (1/2, 1)$,

$$E_{\mu_\lambda}(\alpha) \neq \emptyset \text{ and } \dim_H E\mu_\lambda(\alpha) = \tau^*_{\mu_\lambda}(\alpha)$$

  for those $\alpha = \tau'_{\mu_\lambda}(q)$, $q > 1$, provided that $\tau'_{\mu_\lambda}(q)$ exists at q. The result holds for all self-conformal measures **with overlaps**. (F., 2007)

- **Key idea**: Show that for any $q > 1$, $\exists$ a measure ν_q such that

$$\nu_q(B_r(x)) \preceq r^{-\tau_\mu(q)} \mu_\lambda(B_{16r}(x))^q.$$

  (Inspired from works of Peres-Solomyak (2000), and Brown-Michon-Peyriere (1992)) .
  Then apply some large-deviation like arguments as in Brown-Michon-Peyriere (1992), Ben Nasr (1994) and Testud (2006).

- In the case that λ^{-1} is a Salem number, the condition $q > 1$ can be relaxed to $q > 0$. (F., 2007)

- However, it still remains open whether τ_{μ_λ} is differentiable over $(0, \infty)$ for each λ. Although by concavity τ_{μ_λ} has at most countably many non-differentiable points, no much information can be provided for the range $\{\alpha : \ \alpha = \tau'_{\mu_\lambda}(q) \text{ for some } q > 0\}$.

- **Solomyak's Question**: Does the range of local dimensions of μ_λ contain an interval?

Our main results

Theorem (F., preprint)

Let $\lambda \in (1/2, 1)$ so that λ^{-1} is a Salem number. Then

(i) $E_{\mu_\lambda}(\alpha) \neq \emptyset$ **if** $\alpha \in [\tau'_{\nu_\lambda}(+\infty), \tau'_{\mu_\lambda}(0+)]$, *where* $\tau'_{\mu_\lambda}(+\infty) := \lim_{q\to+\infty} \tau_{\mu_\lambda}(q)/q$, *and* $\tau'_{\mu_\lambda}(0+)$ *denotes the right derivative of* τ_{μ_λ} *at* 0.

(ii) *For any* $\alpha \in [\tau'_{\nu_\lambda}(+\infty), \tau'_{\mu_\lambda}(0+)]$,

$$\dim_H E_{\mu_\lambda}(\alpha) = \tau^*_{\mu_\lambda}(\alpha) := \inf\{\alpha q - \tau_{\mu_\lambda}(q) : q \in \mathbb{R}\}.$$

Theorem (F., preprint)

For $n \geq 4$, let β_n be the largest real root of the polynomials $x^n - x^{n-1} - \cdots - x + 1$, and let $\lambda_n = \beta_n^{-1}$. Then for $\lambda = \lambda_n$, $\tau'_{\nu_\lambda}(+\infty) < 1 \leq \tau'_{\mu_\lambda}(0+)$; and hence **the range of local dimensions of μ_λ contains a non-degenerate interval**.

Sketched proof

1. (**Garsia's Lemma (1962)**): Let β be a Salem number. Then $\exists$ a polynomial $f(x)$ such that for any $\epsilon_1, \ldots \epsilon_k \in \{0, 1, -1\}$,

$$\left| \sum_{n=1}^{k} \epsilon_n \beta^n \right| > \frac{1}{f(k)}$$

if $\sum_{n=1}^{k} \epsilon_n \beta^n \neq 0$.

2. Assume that λ^{-1} is a Salem number in $(1,2)$. For $n \in \mathbb{N}$, denote

$$t_n = \sup_{x\in\mathbb{R}} \#\{S_{i_1\dots i_n} : \ i_1 \dots i_n \in \{1,2\}^n, \ S_{i_1\dots i_n}(K)\cap[x-\lambda^n, x+\lambda^n] \neq \emptyset\},$$

where S_1, S_2 are given as in (1), $S_{i_1\dots i_n} := S_{i_1} \circ \cdots \circ S_{i_n}$ and $K := [-\frac{1}{1-\lambda}, \frac{1}{1-\lambda}]$ is the attractor of $\{S_1, S_2\}$. By Garsia's Lemma,

$$\lim_{n\to\infty} \frac{\log t_n}{n} = 0.$$

Key step

3. **Local box-counting principle For Salem case**:

Given $n \in \mathbb{N}$, $x \in \mathbb{R}$ with $\mu(B_{2^{-n-1}}(x)) > 0$. Let $q > 0$ so that $\alpha = \tau_\mu'(q)$ exists.

Then when m is suitably large, $m = o(n)$, which can be controlled delicately by n, q and $\mu(B_{2^{-n}}(x))/\mu_\lambda(B_{2^{-n-1}}(x)))$, there exist

$$N \succeq 2^{m\tau_\mu^*(\alpha)}$$

many disjoint balls $B_{2^{-n-m}}(x_i)$, $i = 1, \ldots, N$, contained in $B_{2^{-n}}(x)$ such that

$$\frac{\mu(B_{2^{-n-m}}(x_i))}{\mu(B_{2^{-n}}(x))} \sim 2^{-m\alpha},$$

and

$$\frac{\mu(B_{2^{-n-m+1}}(x_i))}{\mu(B_{2^{-n-m-1}}(x_i))} \le C$$

where C is a constant independent of n, m.

Comparison

Classical box-counting principle

For any measure μ, let $q > 0$ so that $\alpha = \tau'_\mu(q)$ exists and let $k \in \mathbb{N}$. Then there exists a sequence $r_n \downarrow 0$ such that for each n, there are $N_n \succeq r_n^{-\tau^*_\mu(\alpha)}$ many disjoint balls $B_{r_n}(x_i)$, so that

$$\mu(B_{r_n}(x_i)) \sim r_n^{\alpha},$$

4. **Moran construction**

Applying this local box-counting principle, for any $\alpha \in [\tau'_{\nu_\lambda}(+\infty), \tau'_{\mu_\lambda}(0+)]$, we give a delicate construction of a Cantor-type subset of $E_{\mu_\lambda}(\alpha)$ with **Moran structure** such that its Hausdorff dimension is greater or equal to $\tau^*_{\mu_\lambda}(\alpha)$. $\square$

Question: Since Bernoulli convolution associated with Salem numbers may have a rich multifractal structure, can we conclude that they are singular?

Absolutely self-similar measures with non-trivial multifractal structures

Theorem (F., preprint)

For $\lambda, u \in (0,1)$, *let* $\Phi_{\lambda,u} := \{S_i\}_{i=1}^3$ *be the IFS on* $\mathbb{R}$ *given by*

$$S_1(x) = \lambda x, \quad S_2(x) = \lambda x + u, \qquad S_3(x) = \lambda x + 1.$$

Let $\mu_{\lambda,u}$ *be the self-similar measure associated with* $\Phi_{\lambda,u}$ *and the probability vector* $\{1/4, 5/12, 1/3\}$, *i.e.,* $\mu = \mu_{\lambda,u}$ *satisfies*

$$\mu = \frac{1}{4}\mu \circ S_1^{-1} + \frac{5}{12}\mu \circ S_2^{-1} + \frac{1}{3}\mu \circ S_3^{-1}.$$

Then for $\mathcal{L}^2$*-a.e.* $(\lambda, u) \in (0.3405, 0.3439) \times (1/3, 1/2)$, $\mu_{\lambda,u}$ *is absolutely continuous, and the range of local dimensions of* $\mu_{\lambda,u}$ *contains a non-degenerate interval, on which the multifractal formalism for* $\mu_{\lambda,u}$ *is valid.*

Idea

Applying a result of **Falconer (1999, Nonlinearity)**, for $1 < q < 2$, for each $0 < \lambda < 1/2$, and for $\mathcal{L}$-a.e. $u \in (0,1)$,

$$\tau(q,\lambda,u) = \min\left\{ \frac{\log\left((1/4)^q + (5/12)^q + (1/3)^q\right)}{\log\lambda},\ q-1 \right\}.$$

For $0 < \lambda < 0.3438$ and $q > 1.5$,

$$\tau(q,\lambda,u) = \frac{\log\left((1/4)^q + (5/12)^q + (1/3)^q\right)}{\log\lambda} > q-1.$$

By **Feng(2007)**, for every $0 < \lambda < 0.3438$, and $\mathcal{L}$-a.e. $u \in (0,1)$, $\mu_{\lambda,u}$ contains the non-degenerate interval $\{\frac{d\tau(q,\lambda,u)}{dq} :\ 1.5 < q < 2\}$, on which the multifractal formalism for $\mu_{\lambda,u}$ is valid.
Absolute continuity comes from a general result by **Peres and Solomyak (1998, TAMS)**.

A recent result of Jordan, Shmerkin and Solomyak on Biased Bernoulli convolutions

For each $\lambda \in (1/2, \gamma)$, where $\gamma \approx 0.554958$ is the root of $1 = x^{-1} + \sum_{n=1}^{\infty} x^{-2n}$, and $p \in (0, 1/2)$, the biased Bernoulli convolution ν_λ^p (which is the the infinite convolution product of the distributions $p\delta_{-\lambda^n} + (1-p)\delta_{\lambda^n}$) always contain a non-trivial interval in the range of its local dimensions.

It is unknown whether or not the multifractal formalism holds for ν_λ^p on this interval.

Open questions for self-similar measures with overlaps

- Is $\tau_\mu(q)$ always differentiable over $(0, \infty)$?
- Is the MF valid when $0 < q < 1$?

D. J. Feng, Multifractal analysis of Bernoulli convolutions associated with Salem numbers. Preprint. (available on www.math.cuhk.edu.hk/$\sim$ djfeng)

Thank you!!!