The Hausdorff dimension of graphs of prevalent continuous functions

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joint work with James T. Hyde

Graphs of continuous functions

Let

$$C[0,1] = \Big\{ f : [0,1] \to \mathbb{R} \mid f \text{ is continuous} \Big\}.$$

This is a Banach space when equipped with the infinity norm, $\| \cdot \|_\infty$.

We define the graph of a function, $f \in C[0,1]$, to be the set

$$G_f = \Big\{ (x, f(x)) \mid x \in [0,1] \Big\} \subset \mathbb{R}^2$$

and are interested in computing its 'dimension'.

Generic dimension of graphs of continuous functions

Over the past 25 years several papers have investigated the question:

What is the 'dimension' of the graph of a 'generic' continuous function?

Clearly, this question can mean different things depending on the definition of the words 'dimension' and 'generic'!

Dimension

There are, of course, several different notions of 'dimension' used to study fractal sets. Some of the most widely used include Hausdorff dimension, packing dimension, box-counting dimension and modified box dimension. These are related in the following way:

$$\dim_{\mathrm{P}} F = \overline{\dim}_{\mathrm{MB}} F$$

$$\dim_{\mathrm{H}} F \leqslant \underline{\dim}_{\mathrm{MB}} F \leqslant \dim_{\mathrm{P}} F, \qquad \overline{\dim}_{\mathrm{MB}} F \leqslant \overline{\dim}_{\mathrm{B}} F$$

$$\underline{\dim}_{\mathrm{MB}} F \leqslant \underline{\dim}_{\mathrm{B}} F \leqslant \overline{\dim}_{\mathrm{B}} F$$

How should we define ‘generic’?

In mathematics one is often interested in making statements about a ‘generic’ member of some family. (*Almost all* real numbers are *normal*, for example.) It is therefore important to develop a rigorous framework in which a sensible definition of ‘generic’ can be given. We will focus on two major approaches to this problem:

(1) Prevalence;

(2) Typicality.

Prevalence: a measure theoretic approach

Definition

Let X be a completely metrizable topological vector space. A Borel set $F \subseteq X$ is prevalent if there exists a Borel measure μ on X and a compact set $K \subseteq X$ such that $0 < \mu(K) < \infty$ and

$$\mu\Big(X \setminus F + x\Big) = 0$$

for all $x \in X$.

A non-Borel set $F \subseteq X$ is prevalent if it contains a prevalent Borel set and the complement of a prevalent set is called a shy set.

Prevalence: an extension of 'Lebesgue almost all' to infinite dimensional spaces

Prevalence was first introduced in the general setting of abelian Polish groups by J. P. R. Christensen in the 1970s and later rediscovered by Hunt, Sauer and Yorke in 1992. The importance of prevalence is that it extends the notion of 'Lebesgue almost all' to infinite dimensional spaces where there is no Lebesgue measure. It satisfies many of the natural properties one would want from a definition of 'generic'. For example:

(1) A superset of a prevalent set is prevalent;

(2) Prevalence is translation invariant;

(3) A countable intersection of prevalent sets is prevalent;

(4) In *finite* dimensional vector spaces prevalent sets are precisely the sets with full Lebesgue measure.

Typicality: a topological approach

Definition

Let X be a complete metric space. A set M is called meagre if it can be written as a countable union of nowhere dense sets. A property is called typical if the set of points which do not have the property is meagre.

Perhaps surprisingly, typicality often completely disagrees with the measure theoretic approach to describing generic behaviour. For example, a typical real number is not normal.

Dimensions of typical graphs

The question stated previously has been completely answered in the 'typicality' case.

Theorem

A typical function $f \in C[0,1]$ satisfies:

$$\dim_{\mathrm{H}} G_f = \underline{\dim}_{\mathrm{MB}} G_f = \underline{\dim}_{\mathrm{B}} G_f = 1 \; < \; 2 = \dim_{\mathrm{P}} G_f = \overline{\dim}_{\mathrm{MB}} G_f = \overline{\dim}_{\mathrm{B}} G_f .$$

Proof.

In 1988 it was shown by Humke and Petruska that the graph of a typical continuous function has packing dimension 2 and in 2010 it was shown by Hyde, Laschos, Olsen, Petrykiewicz and Shaw that the graph of a typical continuous function has lower box dimension 1. □

Dimensions of prevalent graphs

In the 'prevalence' case the question has been partially answered. In 1997 it was shown by McClure that the packing dimension of the graph of a prevalent continuous function is 2.

More recently, it has been shown by Shaw that the lower box dimension of the graph of a prevalent continuous function is also 2. This result was also obtained independently by Gruslys, Jonušas, Mijovič, Ng, Olsen and Petrykiewicz and Falconer and F.

$$1 \leqslant \dim_{\mathrm{H}} G_f \leqslant \underline{\dim}_{\mathrm{MB}} G_f \leqslant 2 = \underline{\dim}_{\mathrm{B}} G_f = \dim_{\mathrm{P}} G_f = \overline{\dim}_{\mathrm{MB}} G_f = \overline{\dim}_{\mathrm{B}} G_f$$

A short note on the proof

In showing that the lower box dimension of the graph of a prevalent continuous function is 2, the following Lemma is key:

Lemma

For all $f, g \in C[0,1]$ *and for Lebesgue almost all* $\lambda \in \mathbb{R}$ *we have*

$$\underline{\dim}_{\mathrm{B}} G_{f+\lambda g} \geqslant \max\{\underline{\dim}_{\mathrm{B}} G_f, \underline{\dim}_{\mathrm{B}} G_g\}$$

The proof of this uses the Borel-Cantelli Lemma.

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Is it true that for all $f, g \in C[0,1]$ *and for Lebesgue almost all* $\lambda \in \mathbb{R}$ *we have*

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Answer: I don't know!

So we need a different approach!

Main result: a complete answer in the 'prevalence' case

Theorem (F and Hyde)

The set

$$\{f \in C[0,1] \mid \dim_{\mathrm{H}} G_f = 2\}$$

is a prevalent subset of $C[0,1]$ *from which it follows that a prevalent function* $f \in C[0,1]$ *satisfies:*

$$\dim_{\mathrm{H}} G_f = \underline{\dim}_{\mathrm{MB}} G_f = \underline{\dim}_{\mathrm{B}} G_f = \dim_{\mathrm{P}} G_f = \overline{\dim}_{\mathrm{MB}} G_f = \overline{\dim}_{\mathrm{B}} G_f = 2.$$

This result should be compared with the result in the 'typicality' case. In particular, the Hausdorff dimension of the graph of a *typical* continuous function and that of a *prevalent* continuous function are as different as possible.

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