Local Universality of Repulsive Particle Systems and Random Matrices

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joint with M.Venker,

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Workshop "Random Matrices and their Applications"

Télécom Paristech October 9, 2012

Topics

- Local Correlation Statistics for Repulsive Systems:
  GUE-Limits (G.- M.Venker)

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- Local Correlation Statistics for Repulsive Systems:

  β-Ensemble Universality (M.Venker)

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- Local Correlation Statistics for Repulsive Systems:
  β-Ensemble Universality (M.Venker)
- Wigner's Semicircular Law for Martingale Ensembles
  (G.- A.Naumov and A.Tikhomirov)

Empirical evidence for ”repulsive particle systems” in $\mathbb{R}$

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All these particle systems show the phenomenon of repulsion.

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Global Marginal Distributions

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$$= \det\left[\frac{\sin\left(\pi(t_i - t_j)\right)}{\pi(t_i - t_j)}\right]_{1\leq i,j\leq k}.$$

Extensions to β-Ensembles

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We conjecture that $P^{\varphi,\beta}_{N,Q}$ has the same bulk local k-correlation, say ρ^k_β, as the Gaussian-β ensemble

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Theorem (Venker 2012)

Write $\varphi(x) := |x|^\beta \exp\{h\}$, h *real analytic and even Schwartz function,* $\alpha^h \geq 0$ *s. th. for all real analytic, strongly convex and even Q with* $\alpha_Q > \alpha^h$*:*

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where $a \in \text{supp}(\mu^h_{Q,\beta}(a))^\circ$. For $h = 0$ and $Q = x^2$, the limit $N \to \infty$ exists for $Q(x) = G(x) := x^2$ and $h = 0$ by Valko-Virag (09).

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Local correlation limits using relaxation flow methods of Bourgade, Erdős, B. Schlein, and H.-T. Yau (2011,2012) instead of potential theory.

Proof: Simple Example

$h(x) := -x^2$ and $\gamma > 0$

$$P^{\gamma}_{N,\alpha}(x) \quad := \quad Z^{-1}_{N,\alpha,\gamma} P^{GUE}_{N,\alpha}(x) \exp\{\gamma \sum_{i<j} (x_i - x_j)^2\},$$

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$$\sum_{i<j} h(x_i - x_j) = \sum_{i<j} \tilde{h}(x_i - x_j) + N \sum_i \int h(x_i - s) d\mu_Q^h(s) + const.$$

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- Claim: Correlation-Fct. of $P_{N,Q}^h$ equivalent to $P_{N,V}$ as $N \to \infty$, where

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$\mathrm{E}\, X_{jk} = 0$ and $\mathrm{E}\, X_{jk}^2 = \sigma_{jk}^2$, $\quad X_{jk} = X_{kj}$ for $1 \leq j < k < \infty$.

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σ-algebras

$$\mathfrak{F}^{(i,j)} := \sigma\{X_{kl} : 1 \leq k \leq l \leq n, (k,l) \neq (i,j)\}, \quad 1 \leq i \leq j \leq n.$$

Semicircle Law for Martingale Ensembles

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Marcenko-Pastur laws for martingale ensembles: G.-Tikhomirov (2004/6), Adamczak (2011)

Counterexamples I

$$\mathbf{X}_n = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{D} \end{pmatrix}, \qquad n = 2m, \quad m = 500 \quad \text{even}$$

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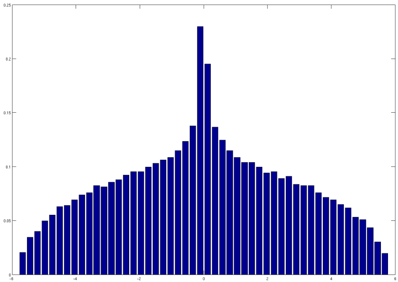

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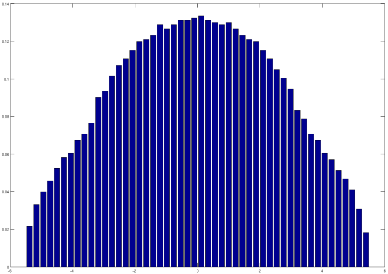

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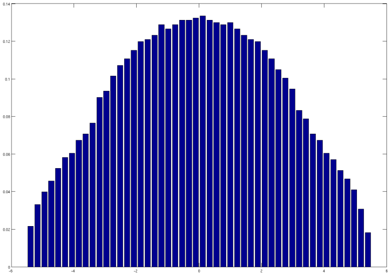

Can be proved via asymptotic freeness of blocks or Lenczewski (arxiv 2012).

Steps of Proof

- Lindeberg-type universality:
  Replacing X_{ij} by Gaussian Y_{ij} using Stieltjes-transforms and conditional moments
- Graph summation using moment methods for non identical Gaussian entries

Thank You!