Concentration phenomena in high dimensional geometry.

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Université Paris-Est Marne-la-Vallée

Workshop ”Random Matrices and their Applications”.
October 2012

Plan.

First part.

• Log-concave measures : a basic concept in probability and geometry.

• Some questions still of interest :

1) Approximation of the covariance matrix

2) The spectral gap inequality : conjecture of Kannan, Lovász and Simonovits

3) The variance conjecture (a particular case of the previous one) and concentration of mass

Second part.

• Another general case : s-concave measures for $s < 0$.

• New results about the concentration of mass.

Log-concave measures.

Let $f : \mathbb{R}^n \to \mathbb{R}+$ such that $\forall x, y \in \mathbb{R}^n, \forall \theta \in [0, 1]$,

$$f((1-\theta)x + \theta y) \geq f(x)^{1-\theta} f(y)^{\theta}$$

A measure with density $f \in L_1^{\mathrm{loc}}$ is said to be log-concave and satisfies $\forall A, B \subset \mathbb{R}^n, \forall \theta \in [0, 1]$,

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Classical examples :
1) Probabilistic : $f(x) = \exp(-|x|_2^2), f(x) = \exp(-|x|_1)$
2) Geometric : $f(x) = 1_K(x)$ where K is a convex body.

Convex geometry - Log-concave measures.

K. Ball

Logarithmically concave functions and sections of convex sets in $\mathbb{R}^n$. Studia Math. 88 (1988), no. 1, 69–84

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R. Kannan, L. Lovász, M. Simonovits
Isoperimetric problems for convex bodies and a localization lemma. Discrete Comput. Geom. 13 (1995), no. 3-4, 541–559.

Random walks and an $O^*(n^5)$ volume algorithm for convex bodies. Random Structures Algorithms 11 (1997), no. 1, 1–50.

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Randomization - Given ε and η, Dyer-Frieze-Kannan('89) established randomized algorithms returning a non-negative number ζ such that

$$(1-\varepsilon)\zeta < \mathrm{Vol}\, K < (1+\varepsilon)\zeta$$

with probability at least $1-\eta$. The running time of the algorithm is polynomial in n, $1/\varepsilon$ and $\log(1/\eta)$.

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The number of oracle calls is a random variable and the bound is for example on its expected value.

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The randomized algorithm proposed by Kannan, Lovász and Simonovits improves significantly the polynomial dependence.

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where $d \leq n^{const}$.

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- John ('48) : $d \leq n$ (or $d \leq \sqrt{n}$ in the symmetric case).
How to find an algorithm to do so ?

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- Idea : find an algorithm which produces in polynomial time a matrix A such that AK is in an approximate isotropic position.

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Computing the volume - Monte Carlo algorithm, estimates of local conductance.

Conjecture 1 of KLS ('95) : isoperimetric inequality - open !

Approximation of the covariance matrix.

Question of KLS ('97) : let X be a vector uniformly distributed on a convex body K, $X_1, \dots, X_N$ ind. copies of X, what is the smallest N such that

$$\left\| \frac{1}{N} \sum_{j=1}^{N} X_j X_j^\top - \mathbb{E} X X^\top \right\| \leq \varepsilon \left\| \mathbb{E} X X^\top \right\|$$

$\| \cdot \|$ is the operator norm

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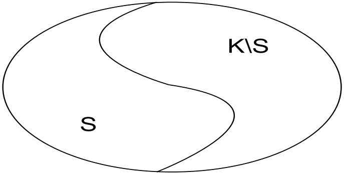
Assume $\mathbb{E} X X^\top = \mathrm{Id}$, you want to control the smallest and the largest singular values.

$$1 - \varepsilon \leq \lambda_{min} \left(\frac{1}{N} \sum_{j=1}^{N} X_j X_j^\top \right) \leq \lambda_{max} \left(\frac{1}{N} \sum_{j=1}^{N} X_j X_j^\top \right) \leq 1 + \varepsilon$$

KLS n^2/ε^2, Bourgain $n \log^3 n/\varepsilon^2$, ... Rudelson, Guédon, Paouris, Aubrun, Giannopoulos
ALPT ('10) n/ε^2 : for general log-concave vectors

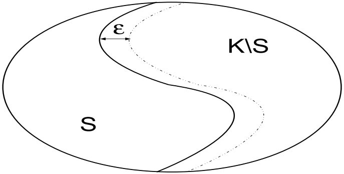
Isoperimetric problem.



Isoperimetric problem.

Define

$$\mu^+(S) = \liminf_{\varepsilon \to 0} \frac{\mu(S + \varepsilon B_2^n) - \mu(S)}{\varepsilon}$$



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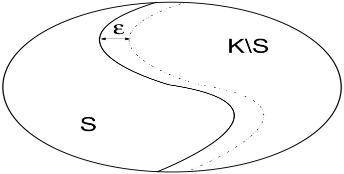

Question. Find the largest h such that

$$\forall\, S \subset K,\ \mu^+(S) \geq\ h\, \mu(S)(1 - \mu(S)) \quad ?$$

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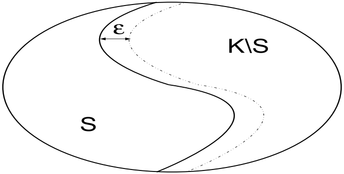

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The probability $d\mu(x) = f(x)dx$ is log-concave isotropic.

Poincaré type inequality. For every regular function F,

$$h^2\ \mathrm{Var}_{\,\mu} F \leq \int |\nabla F(x)|_2^2\, f(x)dx.$$

The conjecture is that h is a universal constant.

Payne-Weinberger ['50] : $h \ge \frac{c}{\operatorname{diam} K}$.

Kannan, Lovász, Simonovits ['95], Bobkov ['07] :

$$h \ge \frac{c}{\int_K |x - g_K|_2 dx} \qquad h \ge \frac{c}{(\operatorname{Var} |X|_2^2)^{1/4}} .$$

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This conjecture implies :

Strong concentration of the Euclidean norm

$$\mathbb{P}\left(\left||X|_2 - \sqrt{n}\right| \geq t\sqrt{n}\right) \leq C\, \exp(-c\, t\, \sqrt{n})$$

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Large and medium scales !

Thin shell and central limit theorem

CLT : classical case. $x_1, \ldots, x_n$, n i.i.d random variables,

$$\mathbb{E}x_i^2 = 1, \mathbb{E}x_i = 0, \mathbb{E}x_i^3 = \tau$$

then $\forall \theta \in S^{n-1}$

$$\sup_{t \in \mathbb{R}} \left| \mathbb{P}\left(\sum_{i=1}^{n} \theta_i x_i \leq t \right) - \int_{-\infty}^{t} e^{-u^2/2} \frac{du}{\sqrt{2\pi}} \right| \leq \tau |\theta|_4^2 = \frac{\tau}{\sqrt{n}}.$$

Thin shell and central limit theorem

Question. [Ball '97], [Brehm-Voigt '98] Let K be an isotropic convex body, find a direction $\theta \in S^{n-1}$ such that

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Conjecture. [Anttila-Ball-Perissinaki '03]

Thin shell conjecture : $\forall n, \exists \varepsilon_n$ such that for every random vector uniformly distributed in an isotropic convex body

$$\mathbb{P}\left(\left| \frac{|X|_2}{\sqrt{n}} - 1 \right| \geq \varepsilon_n \right) \leq \varepsilon_n$$

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Theorem[ABP]. Thin shell $\Rightarrow$ CLT

Concentration of the volume in a Euclidean ball - Large and small scale.

The log-concave case

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In isotropic position, $\mathbb{E}|X|_2^2 = n$ and by classical log-concavity property (cf Borell)

$$\forall t \geq 1, \quad \mathbb{P}\{|X|_2 \geq c\, t\, \sqrt{n}\} \leq 2\, e^{-c\, t}.$$

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Concentration of the volume in a Euclidean ball - Medium scale.

Theorem. Klartag['07] [Fleury-Guédon-Paouris '07]
Let X be a log-concave isotropic vector

$$\forall t > 0, \quad \mathbb{P}\left(\left||X|_2 - \sqrt{n}\right| \geq t\sqrt{n}\right) \leq 2\, e^{-c\sqrt{t}(\log n)^c}.$$

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Theorem [Guédon-Milman '11]

$$\forall t \geq 0, \quad \mathbb{P}\left(\left||X|_2 - \sqrt{n}\right| \geq t\sqrt{n}\right) \leq C \exp(-c\sqrt{n}\ \min(t^3, t))$$

$$\mathrm{Var}\,|X|_2^2 \leq C\, n^{5/3} \quad \textit{and} \quad h \geq c\, n^{-5/12}$$

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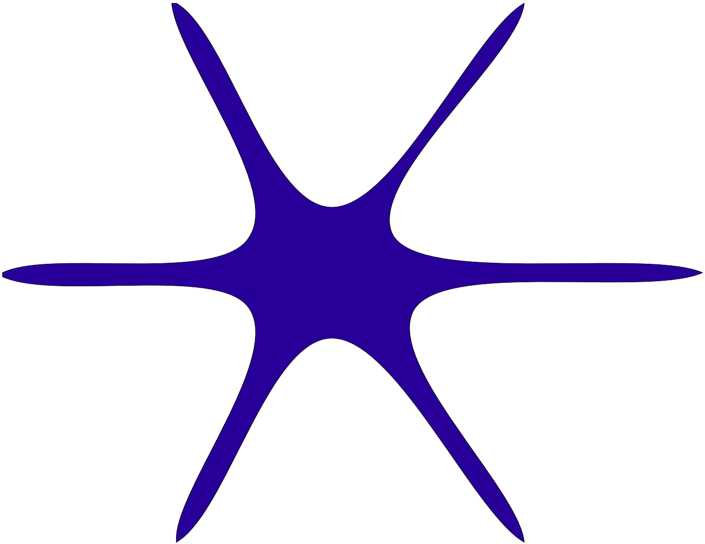
Variance conjecture : $\mathrm{Var}\,|X|_2 \leq C$ or $\mathrm{Var}\,|X|_2^2 \leq Cn$

Pictures - Intuition in high dimension.



convex body in ”isotropic position”.

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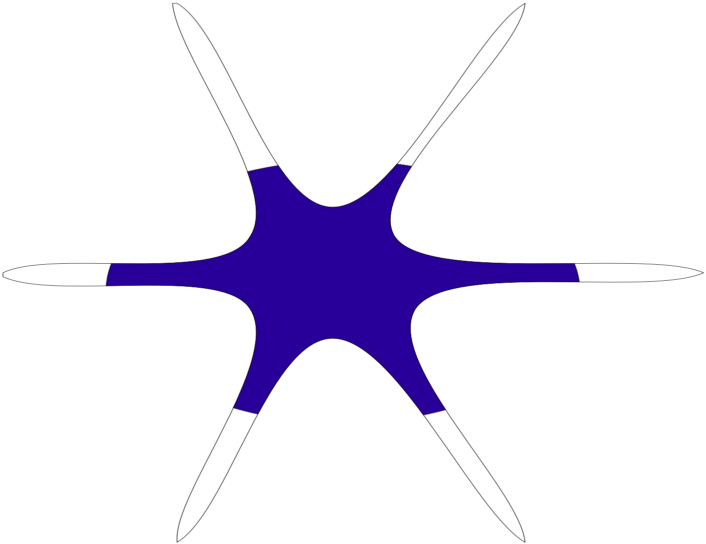

intersection with a ball of radius $\sqrt{n}$.

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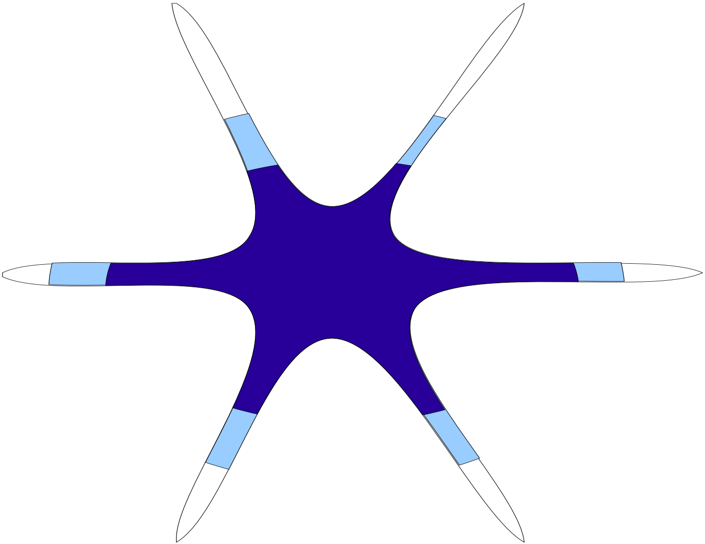

volume inside a ball of radius $100\sqrt{n}$

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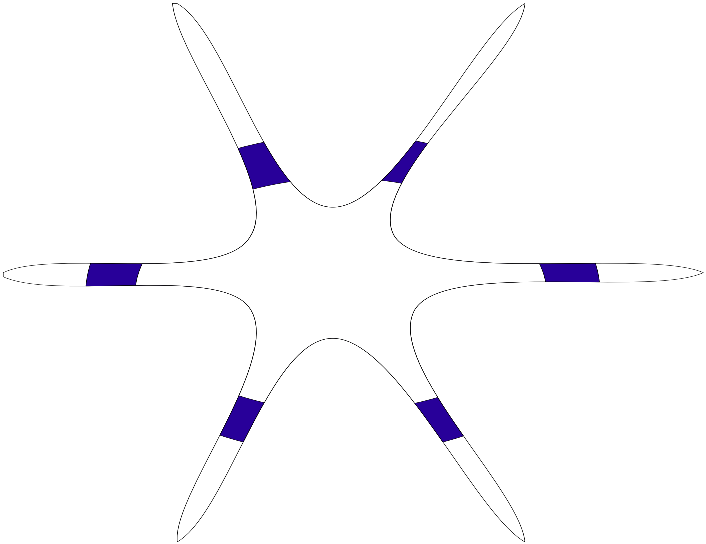

volume inside a shell of width $\sqrt{n}/n^{1/6}$

Concentration of the mass in a Euclidean ball or shell

$\Leftrightarrow$

Behavior of $\left(\mathbb{E}|X|_2^p\right)^{1/p}$ for some values of p.

Concentration of the mass in a Euclidean ball or shell

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Behavior of $\left(\mathbb{E}|X|_2^p\right)^{1/p}$ for some values of p.

• X log-concave random vector. Paouris Theorem (large deviation) may be written as (ALLOPT '12)

$$\forall p \geq 1, \quad \left(\mathbb{E}|X|_2^p\right)^{1/p} \leq C\, \mathbb{E}|X|_2 + c\, \sigma_p(X) \qquad (\star)$$

where $\sigma_p(X) = \sup_{|z|_2 \leq 1} \left(\mathbb{E}\langle z, X\rangle^p\right)^{1/p}$.

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Hence $\quad \forall p \geq 1, \quad \left(\mathbb{E}|X|_2^p\right)^{1/p} \leq C\sqrt{n} + cp$

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Take $p = t\sqrt{n}$, Markov gives

$$\forall t \geq 1, \quad \mathbb{P}\left(|X|_2 \geq t\sqrt{n}\right) \leq e^{-c\,t\sqrt{n}}.$$

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• Small Ball Estimates of Paouris - Negative moments.

• Variance conjecture - slightly more, cf KLS. In isotropic position,

$$\forall p \in [2, c\sqrt{n}], \quad \left(\mathbb{E}|X|_2^p\right)^{1/p} \leq \sqrt{n} + c\,\frac{p}{\sqrt{n}} = \left(\mathbb{E}|X|_2^2\right)^{1/2}\left(1 + \frac{c\,p}{n}\right).$$

• In view of $(\star)$, more tractable conjecture :

$$\forall p \geq 1, \quad \left(\mathbb{E}|X|_2^p\right)^{1/p} \leq \mathbb{E}|X|_2 + c\, \sigma_p(X)$$

Other probabilistic questions.

For which random vector do we have that for any norm,

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Paouris Theorem tells that it is true for log-concave and the Euclidean norm !

New class of random vectors

The hypothesis $H(p,\lambda)$:
Let $p > 0$, $m = \lceil p \rceil$, and $\lambda \geq 1$. A random vector X in E satisfies the assumption $H(p,\lambda)$ if for every linear mapping $A : E \to \mathbb{R}^m$ s. t. $Y = AX$ is non-degenerate there exists a gauge $\|\cdot\|$ on $\mathbb{R}^m$ s. t. $\mathbb{E}\|Y\| < \infty$ and

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Results. (AGLLOPT* '12)

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Theorem 1 *Let $p>0$ and $\lambda\ge 1$. If a random vector X satisfies $H(p,\lambda)$ then*

$$\left(\mathbb{E}|X|_2^p\right)^{1/p}\le c\left(\lambda\mathbb{E}|X|_2+\sigma_p(X)\right)$$

where c is a universal constant.

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Proof : X random vector in E, $m = \lceil p \rceil$, $\lambda \geq 1$, $A : E \to \mathbb{R}^m$

Gaussian Concentration. G standard Gaussian vector

$$(\mathbb{E}_G \mathbb{E}_X \langle G, X \rangle^p)^{1/p} \leq \mathbb{E}_G (\mathbb{E}_X \langle G, X \rangle^p)^{1/p} + c\,\sqrt{p}\,\sigma_p(X)$$

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$$\|z\| = (\mathbb{E}_X \langle z, X \rangle^p)^{1/p}$$

is the dual norm of Z_p bodies, at the heart of all proofs.

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$$\lesssim \frac{1}{\sqrt{p}} \mathbb{E}_A\, \lambda\, \mathbb{E}_X |AX|_2 + \sigma_p(X) \lesssim \lambda\, \mathbb{E}|X|_2 + \sigma_p(X)$$

s-concave random vectors, $s < 0$

Convex measures : definition

Let $s < 1/n$. A probability Borel measure μ on $\mathbb{R}^n$ is called s-concave if $\forall A, B \subset \mathbb{R}^n, \forall \theta \in [0,1]$,

$$\mu((1-\theta)A + \theta B) \geq ((1-\theta)\mu(A)^s + \theta\mu(B)^s)^{1/s}$$

whenever $\mu(A)\mu(B) > 0$.

For $s = 0$, this corresponds to log-concave measures.

The class of s-concave measures was introduced and studied by Borell in the 70's. A s-concave probability ($s \leq 0$) is supported on some convex subset of an affine subspace where it has a density.

s-concave random vectors, $s < 0$

Convex measures : properties

Let $s = -1/r$.

When the support generates the whole space, a convex measure has a density g which has the form

$$g = f^{-\beta} \quad \text{with} \quad \beta = n + r$$

and f is a positive convex function on $\mathbb{R}^n$. (Borell).

Example :

$$g(x) = c(1 + \|x\|)^{-n-r}, r > 0.$$

- A log-concave prob is $(-1/r)$-concave for any $r > 0$
- The linear image of a $(-1/r)$-concave vector is also $(-1/r)$-concave.
- The Euclidean norm of a $(-1/r)$-concave random vector has moments of order $0 < p < r$.

Convex measures and $H(p, \lambda)$

Theorem 2. *Let $r \geq 2$ and X be a $(-1/r)$-concave random vector. Then for every $0 < p < r/2$, X satisfies the assumption $H(p, C)$, C being a universal constant.*

Theorem 3. *Let $r \geq 2$ and X be a $(-1/r)$-concave random vector. Then for every $0 < p < r/2$,*

$$(\mathbb{E}|X|_2^p)^{1/p} \leq C(\mathbb{E}|X|_2 + \sigma_p(X)).$$

Convex measures. Concentration of $|X|_2$

Corollary. *Let $r \geq 2$ and X be a $(-1/r)$-concave random vector. Then for every $t > 0$,*

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Corollary. *Let $r \geq \log n$ and X be a $(-1/r)$-concave isotropic random vector. Let $X_1, \ldots, X_N$ be independent copies of X. Then for every $\varepsilon \in (0, 1)$ and every $N \geq C(\varepsilon)n$, one has*

$$\mathbb{E}\left\|\frac{1}{N}\sum_{i=1}^{N} X_i X_i^\top - I\right\| \leq \varepsilon.$$

THANK YOU