Multifractal Analysis of the Divergence of Fourier Series

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- Carleson Hunt (1967) Always true if $f \in L^p(\mathbb{T})$, $p > 1$

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$$\|S_nf\|_\infty \leq Cn^{1/p}\|S_nf\|_p \leq Cn^{1/p}\|f\|_p \ .$$

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Let $\beta \in [0, 1/p]$. What is the size of the set of points x such that $|S_n f(x)| \approx n^\beta$ when $n \to +\infty$?

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Question

Let $\beta \in [0,1]$. What is the size of the set of points x such that $|S_n f(x)| \approx (\log n)^\beta$ when $n \to +\infty$?

Aubry's result

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$$\mathcal{E}(\beta, f) = \left\{ x \in \mathbb{T};\ \limsup_{n \to +\infty} n^{-\beta} |S_n f(x)| > 0 \right\}.$$

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Suppose $p > 1$ and $f \in L^p(\mathbb{T})$. Then

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Carleson Hunt maximal inequality :

$$\|S^* f\|_p \leq C \|f\|_p$$

where $S^* f(x) = \sup_n |S_n f(x)|$.

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Question

Can we construct a function $f \in L^p(\mathbb{T})$ for which lower bounds of $\dim_{\mathcal{H}} \left(\mathcal{E}(\beta, f) \right)$ are valid for all $\beta \in [0, 1/p]$?

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Multifractal analysis :

$$\beta \mapsto \dim_{\mathcal{H}}\left(E(\beta, f)\right)\ .$$

Multifractal behavior of $S_n f$

Of course, $E(\beta, f) \subset \bigcap_{\gamma<\beta} \mathcal{E}(\gamma, f)$, so that

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- "quasi-all" is related to the Baire category theorem.
- For such f we also have $\dim_{\mathcal{H}} \left(\mathcal{E}(\beta, f)\right) = 1 - \beta p$.

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$$\beta = \frac{1}{p}\left(1 - \frac{1}{\alpha}\right) \Rightarrow \frac{1}{\alpha} = 1 - \beta p$$

What about $p = 1$?

Theorem (Bayart, H., 2011)

For quasi-all functions $f \in L^1(\mathbb{T})$,

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- Carleson Hunt maximal inequality is false in this context
- We can prove the following sufficient estimation :

$$\int_{\mathbb{T}} \sup_n \left| \frac{S_n f(x)}{(\log n)^{1+\varepsilon}} \right| dx \leq C \|f\|_1 \ .$$

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Level sets :

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Roughly speaking, for all β, $S_n f(x) \approx (\log n)^\beta$ in a set with dimension 1.

The precised Hausdorff dimension

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We say that E has precised Hausdorff dimension (α, β) if α is the Hausdorff dimension of E and

- $\beta = 0$ if $\mathcal{H}^{\phi_{\alpha,t}}(E) = 0$ for every $t \in (0,1)$;
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Natural order : lexicographical order. The bigger is the set, the bigger is the precised Hausdorff dimension.

Precised Theorem

Theorem (Bayart,H., 2010)

For quasi-all functions $f \in \mathcal{C}(\mathbb{T})$, $\quad \forall \beta \in [0,1]$, *the precised Hausdorff dimension of* $F(\beta, f)$ *is* $(1, 1-\beta)$.

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Let $\beta \in (0,1)$ and $f \in \mathcal{C}(\mathbb{T})$.

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- The lemma gives the upper bound for the precised dimension

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Remarks

- Aubry's type result
- The lemma gives the upper bound for the precised dimension
- Tool : $m\big(\{x \in \mathbb{T}\ ;\ S^*f(x) > y\}\big) \leq Ae^{-By/\|f\|_\infty}$

Basic idea to construct divergence points

$$P_N(x) = e_N(x) \sum_{j=1}^{N} \frac{\sin(2\pi j x)}{j},$$

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Similar results are needed in a big set of points x.

Construction of polynomials P

Fact (Kahane Katznelson) : Let $Q(z) = \sum_{k=1}^{n-1} a_k z^k$. Set $P(\theta) = e^{in\theta} \times \Im m Q(e^{i\theta})$.

$$|S_n P(\theta)| = \frac{1}{2} \left| \sum_{k=1}^{n-1} \overline{a_k} e^{i(n-k)\theta} \right| = \frac{1}{2} \left| Q(e^{i\theta}) \right| \; .$$

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The polynomials Q are constructed by taking the logarithm of the holomorphic function

$$f(z) = \frac{1}{k} \sum_{j=0}^{k-1} \frac{1+\varepsilon}{1+\varepsilon - \overline{z_j} z}$$

$z_j = e^{(2i\pi j)/k}$.

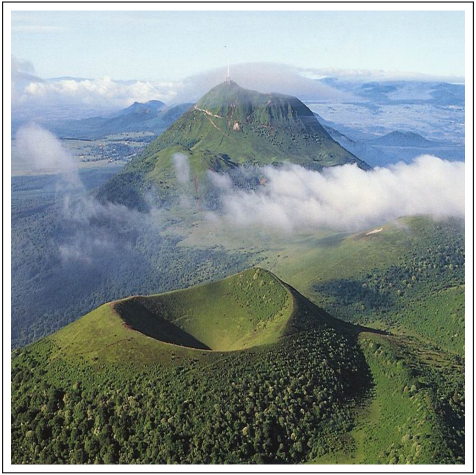
The main lemma

Lemma

Let $\beta \in (0,1)$. Then there exist k and n as large as we want and a trigonometric polynomial P with spectrum contained in $[0, 2n-1]$ such that

- $\forall x \in \mathbb{T}, \quad |P(x)| \leq 1$
- $\forall x \in I_k^\beta, \quad \log |S_n P(x)| \geq \beta \log \log n$ *for any* $x \in I_k^\beta$,

where $I_k^\beta = \bigcup_{j=0}^{k-1} \left[\frac{j}{k} - \frac{1}{2k \exp\left((\log k)^\beta\right)}; \frac{j}{k} + \frac{1}{2k \exp\left((\log k)^\beta\right)} \right]$.



Merci !