Infinite IFS: limit sets and continuity

Martial Hille

HU Berlin

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Outline of the talk

1. Motivation
2. Definition of different limit sets
3. Some results on limit sets
4. Set of accumulation points and relations to limit sets

Based on the recent preprints

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5. The space of self similar IFSs
6. Metric on the space of s.s. IFSs
7. Continuity results

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8. Future prospects

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Motivation: Finite vs. infinite IFS

Let $S := \{S_i \mid i \in I\}$ satisfy OSC, I finite, then $\exists!$ compact $K \neq \emptyset$ s.t.

$$K = \bigcup_{i \in I} S_i(K), \qquad \text{and we have } K = \pi(I^\infty).$$

If I is infinite, this generalises to two different sets, namely the invariant set

$$\pi(I^\infty) = \bigcup_{i \in I} S_i(\pi(I^\infty)),$$

and the closed set

$$\overline{\pi(I^\infty)}.$$

Different limit sets

Definition

Let $\mathbf{S} = \{S_i : X \to X \mid i \in I := \mathbb{N}\}$ be an infinite IFS acting on a compact set $X \subset \mathbb{R}^m$, with $X = \overline{\mathrm{Int}(X)}$ which satisfies OSC.
We then define the following types of limit sets of $\mathbf{S}$.

- The *dynamical limit set* $\mathrm{L}_{\mathrm{dyn}}(\mathbf{S}) := \pi(I^\infty)$.
- The *limit set* $\mathrm{L}(\mathbf{S}) := \overline{\mathrm{L}_{\mathrm{dyn}}(\mathbf{S})}$.
- The *Jørgensen limit set* $\mathrm{L}_{\mathrm{J}}(\mathbf{S}) := \mathrm{L}(\mathbf{S}) \setminus \mathrm{L}_{\mathrm{dyn}}(\mathbf{S})$.

Question

What is the relation between $\dim_{\mathrm{H}} \mathrm{L}_{\mathrm{J}}(\mathbf{S})$ and $\dim_{\mathrm{H}} \mathrm{L}_{\mathrm{dyn}}(\mathbf{S})$?

Some answers

Theorem (Moran '95/ Mauldin-Urbanski'96)

For all self-similar infinite IFS

$$\dim_{\mathrm{H}} \mathrm{L}_{\mathrm{dyn}}(\mathbf{S}) = \inf\{s > 0 \mid \sum_{i \in I} r_i^s \leq 1\}.$$

Theorem (H.)

For every $m \in \mathbb{N}$ and $d, j \in (0, m)$ there exists an infinite (self-similar) IFS $\mathbf{S}$ acting on $\mathbb{R}^m$ such that

$$\dim_{\mathrm{H}} \mathrm{L}_{\mathrm{dyn}}(\mathbf{S}) = d \quad \textit{and} \quad \dim_{\mathrm{H}} \mathrm{L}_{\mathrm{J}}(\mathbf{S}) = j.$$

(Moreover, $\mathbf{S}$ is regular and satisfies SSC.)

Set of accumulation points and limit sets

Definition of the set of accumulation points

$$\mathrm{Acc}(\mathbf{S}) := \bigcup_{J \subset I} \left(\overline{\bigcup_{i \in J} S_i(X)} \setminus \bigcup_{i \in J} S_i(X) \right)$$

Some relations to limit sets

We have

$$\mathrm{L_J}(\mathbf{S}) \subset \mathcal{O}_{\mathbf{S}}(\mathrm{Acc}(\mathbf{S})) \quad \text{and} \quad \mathrm{Acc}(\mathbf{S}) \subset \mathrm{L}(\mathbf{S})$$

and hence

$$\begin{aligned} \dim_{\mathrm{H}} \mathrm{L}(\mathbf{S}) &= \max\{\dim_{\mathrm{H}} \mathrm{L_{dyn}}(\mathbf{S}), \dim_{\mathrm{H}} \mathrm{L_J}(\mathbf{S})\} \\ &= \max\{\dim_{\mathrm{H}} \mathrm{L_{dyn}}(\mathbf{S}), \dim_{\mathrm{H}} \mathrm{Acc}(\mathbf{S})\}. \end{aligned}$$

A priori conditions for $\mathcal{O}_{\mathbf{S}}(\mathrm{Acc}(\mathbf{S})) = \mathrm{L_J}(\mathbf{S})$.

Theorem (H.)

Let **S** *satisfy OSC.*

$$\textit{If} \quad \exists\, \overline{U} \subset \mathrm{Int}\, X \quad \textit{s.t.} \quad S_i(X) \subset \overline{U}, \quad \forall i \in I,$$

then

$$\mathcal{O}_{\mathbf{S}}\left(\mathrm{Acc}(\mathbf{S})\right) = \mathrm{L_J}(\mathbf{S}).$$

Theorem (H. - one dimensional systems)

If **S** *satisfies OSC and if* $X \subset \mathbb{R}$ *consists of finitely many intervals, then*

$$\dim_{\mathrm{H}} \mathrm{L_J}(\mathbf{S}) = \dim_{\mathrm{H}} \mathrm{Acc}(\mathbf{S}).$$

The space of s.s. IFSs (joint work with N.Snigireva [HS])

If **S** satisfies SSC then **S** satisfies OSC, but how does the space of IFS look like?

A first answer: Fix $I \subset \mathbb{N}$ and $X \subset \mathbb{R}^m$ as above. Then define

$$\begin{aligned}
\mathrm{IFS}(X, I) &:= \{\mathbf{S} = \{S_i : X \to X \mid i \in I\}, \mathbf{S} \text{ satisfies OSC}\}, \\
\mathrm{IFS}_{\mathrm{SSC}}(X, I) &:= \{\mathbf{S} \in \mathrm{IFS}(X, I), \mathbf{S} \text{ satisfies SSC}\}, \\
\mathrm{IFS}_{\mathrm{OSC}}(X, I) &:= \mathrm{IFS}(X, I) \setminus \mathrm{IFS}_{\mathrm{SSC}}(X, I).
\end{aligned}$$

Theorem (H.S.)

Under certain conditions on X we have

$$\begin{aligned}
\mathrm{IFS}_{\mathrm{SSC}}(X, I) &= \mathrm{Int}\,\mathrm{IFS}(X, I). \\
\mathrm{IFS}_{\mathrm{OSC}}(X, I) &\subsetneq \partial \mathrm{IFS}(X, I).
\end{aligned}$$

$\partial\mathrm{IFS}(X, I)$ *also contains inhomogeneous IFS, as well as pathological systems.*

Parameter space of $\mathrm{IFS}(X, I)$ [HS]

Parametrisation

Each similarity S_i is of the form

$$S_i : x \mapsto r_i \cdot O_i(x) + b_i \quad \text{with} \quad \begin{array}{l} r_i \in (0,1), \\ O_i \in O(m) \subset \mathrm{GL}(m), \\ b_i \in \mathbb{R}^m. \end{array}$$

With this identification $\quad S_i \quad \in \quad (0,1) \times \mathbb{R}^m \times O(m).$

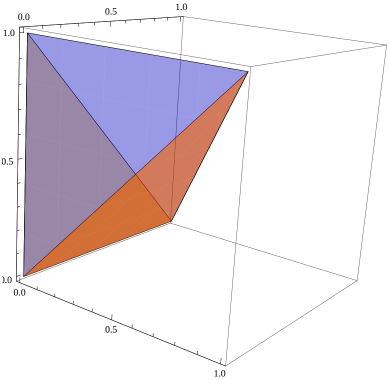
Parameter space of $\mathrm{IFS}(X, I)$

Each $\mathbf{S} \in \mathrm{IFS}(X, I)$ can be identified with an element in

$$((0,1) \times \mathbb{R}^m \times O(m))^I.$$

Example of $\mathrm{IFS}([0,1],\{1,2\})$ [HS]

Let $X := [0,1]$, $I := \{1,2\}$ and fix $O_1 = O_2 = 1$ and fix $b_1 = 0$.



Metric on parameter space of s.s. IFS [HS]

On the space of similarities

$$\{S_i \quad \in \quad (0,1) \times \mathbb{R}^m \times O(m)\}$$

we define a metric d by

$$d(f_1, f_2) := |\log r_1 - \log r_2| + \|b_1 - b_2\| + d(O_1, O_2).$$

d extends to an extended metric on $\mathrm{IFS}(X, I)$ by

$$d_{ex}(\mathbf{S}, \mathbf{T}) := \sum_{i \in I} d(S_i, T_i) \quad \in [0, \infty].$$

$$d(\mathbf{S}, \mathbf{T}) := \frac{d_{ex}(\mathbf{S}, \mathbf{T})}{1 + d_{ex}(\mathbf{S}, \mathbf{T})} \text{ is now a metric.}$$

Continuity results [HS]

Theorem (Continuity (H.S.))

The following maps are continuous w.r.t. d. (and $d_{Hausdorff}$) :

$$\begin{array}{lllllll} \mathbf{S} & \mapsto & \dim_{\mathrm{H}} \mathrm{L}(\mathbf{S}), & \mathbf{S} & \mapsto & \dim_{\mathrm{H}} \mathrm{L}_{\mathrm{dyn}}(\mathbf{S}) \\ \mathbf{S} & \mapsto & \mathrm{L}(\mathbf{S}), & \mathbf{S} & \mapsto & \mathrm{L}_{\mathrm{dyn}}(\mathbf{S}), \\ \mathbf{S} & \mapsto & \mathrm{Acc}(\mathbf{S}) & \multicolumn{3}{l}{(\textit{even locally constant}).} \end{array}$$

If $X \subset \mathbb{R}$ consists of finitely many intervals, then the map

$$\mathbf{S} \mapsto \dim_{\mathrm{H}} \mathrm{L}_{\mathrm{J}}(\mathbf{S}) \quad \textit{is continuous w.r.t. } d.$$

Theorem (Discontinuity (H.S.))

The maps $\mathbf{S} \mapsto \mathrm{L}_{\mathrm{J}}(\mathbf{S})$ and $\mathbf{S} \mapsto \dim_{\mathrm{H}} \mathrm{L}_{\mathrm{J}}(\mathbf{S})$ are in general not continuous.

Future prospects

(i) Continuity results for conformal **S** under (certain) deformations of the seed space X.

(ii) Relation to the modular space of $\mathrm{IFS}(X, I)$.

(iii) Generalisation to (pseudo) graph directed Markov systems.

Thank you!

This talk was based on:

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