1-forms on fractals and harmonic spaces

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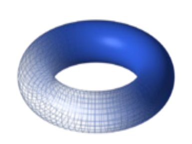
Introduction

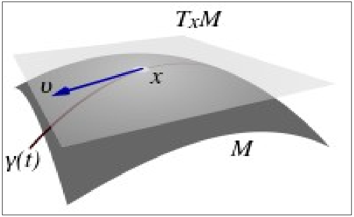
- Consider a smooth scalar function f on a smooth manifold M



- evaluation of differential $df(v)$ says how quickly f *changes if argument changes smoothly with a certain speed* $\|v\|$ *in a certain direction* $\frac{v}{\|v\|}$

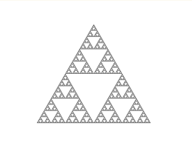
- general differential forms $\omega \in \Omega^*(M)$ tell a lot about the structure of M, more than just functions $f \in \Omega^0(M)$ (curvature, structure equations; topology, etc.)
- 1-forms $\omega \in \Omega^1(M)$ allow to phrase vector equations (in Riemannian case)

- classical definition needs tangent spaces $T_x M$



- many interesting objects ...

- ... and somehow related mathematical toy models ...



- ... do have 'almost no' tangent spaces

- Analysis of functions on (classes of) fractals is relatively well understood (*Goldstein, Kusuoka, Barlow, Bass, Kigami, ...* since late 80's)
- can phrase elliptic

$$\Delta u = f$$

and parabolic

$$\frac{\partial u}{\partial t} = \Delta u$$

PDE.

aims:

- approaches towards useful substitutes for 1-forms, gradients ... on some (classes of) fractals, metric spaces, harmonic spaces
- like to phrase vector valued equations like

$$\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla) u + \nabla p = f \ , \ div\ u = 0$$

- like to see topology and geometry reflected in analysis
- so far: non-classical gradients of *Kusuoka, Kigami, Strichartz, Teplyaev* ... unfortunately quite difficult to use

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rough ideas:

- *approach 1 - if possible:*
  discrete approximations by cell complexes and rescaled limits of discrete 'combinatorial' energy functionals for functions on edges (H. '10 Sierpinski gasket)
- *approach 2 - if there is a given energy functional for functions:*
  'forget about smoothness, use energy' ... (*mainly discussed here*)

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Energy measures

X locally compact Hausdorff topological space, second countable.

Setup 1:

- m σ-finite Radon measure on $\mathcal{B}(X)$, non-zero on non-empty open sets
- $(\mathcal{E}, \mathcal{F})$ *conservative local symmetric Dirichlet form* on $L_2(X, m)$
- $\mathcal{F} = \mathcal{F}_0 \oplus \mathcal{H}$ ... Riesz/Weyl-decomposition with $(\mathcal{E}, \mathcal{F}_0)$ regular ... similarly for restrictions to $U \subset X$ open
- choice of reference measure m is 'unimportant' !
- *Fukushima et. al. '80/'94, Ma/Röckner '92, ...*
- for topo results, require harmonic functions to be continuous

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- *example 1:* $\Omega \subset \mathbb{R}^n$ bounded, $\partial\Omega$ smooth,

$$H^1(\Omega) = H_0^1(\Omega) \oplus \mathcal{H}(\Omega)$$

- *example 2:* $\mathcal{E}$ standard Dirichlet form on Sierpinski gasket,

$$dom(\mathcal{E}) = dom_0(\mathcal{E}) \oplus \text{3-dim. space of harmonic functions}$$

sim. on smaller copies
see Kigami '01, Strichartz '06, ...

- Given $g, h \in \mathcal{F}_0 \cap L_\infty(X, m)$, there is a unique finite signed Borel measure $\nu_{g,h}$ on X such that

$$2 \int_X f d\nu_{g,h} = \mathcal{E}(fg, h) + \mathcal{E}(h, g) - \mathcal{E}(gh, f)$$

  for $f \in \mathcal{F}_0 \cap C_b(X)$.

- $$\mathcal{E}(g, h) = \nu_{g,h}(X)$$

  ... *energy measure*
  LeJan'78, Fukushima et al '80, '94

- $\nu_g := \nu_{g,g} \geq 0$
- *example:* $\Omega \subset \mathbb{R}^n$, $\nu_{g,h} = \nabla g \nabla h \, dx$

Setup 2:

- $(X, \mathcal{H})$ *Brelot harmonic space* carrying a consistent system of symmetric Green's functions
  Brelot, Bauer, Doob, ..., Maeda '80
- for $U \subset X$ open, $\mathcal{H}(U)$ space of continuous harmonic functions
- can similarly define energy measures $\nu_{g,h}$...
  ... $g, h \in \mathcal{D}_{DB}(X) :=$ space of bounded continuous Dirichlet energy finite functions (not just classes)
- *examples:* open Euclidean domains, p.c.f fractals ...

1-forms

- Consider $\mathcal{D}_{DB}(X) \otimes \mathcal{B}_b(X)$ endowed with

$$\langle a \otimes b, c \otimes d \rangle := 2 \int_X bd \, d\nu_{a,c}.$$

- Cipriani/Sauvageot '03 (C^*, vNA, inspired by NCG), '09 (p.c.f. examples)
- Ionescu/Rogers/Teplyaev '11 (finitely ramified, Hodge)
- Cipriani/Guido/Isola/Sauvageot '11 (Sierpinski: Hodge, integration, Čech coho, Abelian covers)
- H.'11 (modified version, harmonic spaces, covers, Hodge; ideal boundaries, Kirchhoff, Čech coho)

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- $\mathring{\Omega}^1(X)$ quotient space after factoring out zero elements, pre-Hilbert

Definition

$\Omega^1(X)$ closure of $\mathring{\Omega}^1(X)$ in $\langle\cdot,\cdot\rangle$ ... *space of* 1*-forms on* X.

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- introduce *module actions:* for $a \otimes b \in \mathcal{D}_{DB}(X) \otimes \mathcal{B}_b(X)$, $c \in \mathcal{D}_{DB}(X)$ and $d \in \mathcal{B}_b(X)$, set

$$c(a \otimes b) := (ca) \otimes b - c \otimes (ab)$$

and

$$(a \otimes b)d := a \otimes (bd).$$

Lemma

Both lines define bounded module actions on $\Omega^1(X)$ that are continuous w.r.t. bounded pointwise convergence.

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- introduce *derivation* $\partial : \mathcal{D}_{DB}(X) \to \Omega^1(X)$ by

$$\partial a = a \otimes \mathbf{1}\ ,\ a \in \mathcal{D}_{DB}(X).$$

- $\|a \otimes \mathbf{1}\|^2 = \nu_a(X) \quad (= \mathcal{E}(a))$
- in metric case consistent with *Cheeger '99*
- *Leibniz rule* holds:

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Open covers (H.'11)

Theorem

$$\Omega^1(X) = clos\ span\ (\{a \otimes \mathbf{1}_V : a \in \mathcal{D}_{DB}(X), V \subset X open\})$$

Proof.

Dynkin class theorem. □

Corollary

$$\Omega^1(X) = \textit{clos span}\ \left(\bigcup \Omega^1(\mathcal{U}) : \mathcal{U} = \{U_\alpha\}_{\alpha\in J}\ \textit{open cover of } X\right)$$

with

$$\Omega^1(\mathcal{U}) = \left\{\sum_{\alpha\in J} a_\alpha \otimes \mathbf{1}_{U_\alpha} : a_\alpha \in \mathcal{D}_{DB}(X)\right\}$$

(by convention: finite linear combinations).

- $\mathcal{G}(X) := \{a \otimes \mathbf{1} : a \in \mathcal{D}_{DB}(X)\}$ *space of exact* 1*-forms*
- $\mathcal{H}^1(X) :=$ *clos span* $\left(\bigcup \mathcal{H}^1(\mathcal{U}) : \mathcal{U} = \{U_\alpha\}_{\alpha \in J}\right.$ open cover of $\left.X\right)$ with

$$\mathcal{H}^1(\mathcal{U}) = \left\{ \sum_{\alpha \in J} h_\alpha \otimes \mathbf{1}_{U_\alpha} : h_\alpha \in \mathcal{D}_{DB}(X) \text{ harmonic on } U_\alpha \text{ and} \right.$$

$$\left. \text{such that } \sum_{\alpha \in J} \nu_{h_\alpha, a}(U_\alpha) = 0 \text{ for any } a \in \mathcal{D}_{DB}(X) \right\}$$

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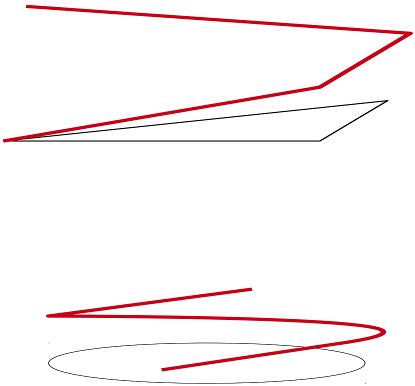

Theorem

$$\Omega^1(X) = \mathcal{G}(X) \oplus \mathcal{H}^1(X) \ ,$$

the sum being orthogonal (Polar decomposition / Hodge).

Proof.

Consider fixed covers $\mathcal{U}$ first. □

investing some potential theory, can simplify and go further:

Theorem

(i) $\mathcal{H}^1(\mathcal{U})$ *non-trivial iff* $\mathcal{U}$ *has a (Čech) cycle.*

(ii) *If* $\mathcal{U}$ *is 'separated', then* $\mathcal{H}^1(\mathcal{U}) \cong \check{H}^1(\mathcal{U})$.

(iii) X *compact, topo one-dimensional, then* $\mathcal{H}^1(X) \cong \check{H}^1(X)$.

Proof.

(one way, not best) Neumann problems, ideal boundaries ('projection onto Čech graph'). □

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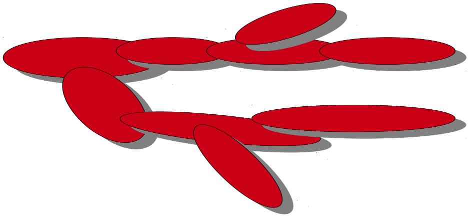

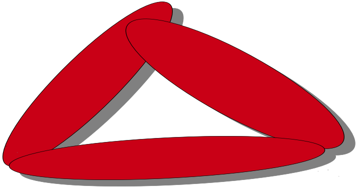

- **justification for nomenclature:** consistency with smooth case, module actions, Leibniz rule, polar decomposition, Čech coho
- *examples:* p.c.f and finitely ramified fractals, metric graphs, one-dimensional manifolds, Sierpinski carpets ...

Combinatorial approach (H.'10), sketch

- $X = SG$ Sierpinski gasket
- approximate SG by two-dim. cell complexes K_n ('cut-out')
- consider vector space $C_1(K_n)$ of 1-chains on K_n
- and associated combinatorial Laplacians

$$\Delta_1 = \partial_{2\to 1}\delta_{1\to 2} + \delta_{0\to 1}\partial_{1\to 0}$$

  and 'natural' scalar products
- to define energies

$$\mathfrak{E}_{K_n}(c) = (c, \Delta_1 c)_{K_n}\ ,\ \ c \in C_1(K_n)$$

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- extend harmonically, respect Hodge, renormalize á la Kigami and get

Theorem

There are an infinite dimensional space $\widetilde{\Omega_1}(SG)$ containig all harmonic elements and a limit energy functional $\mathfrak{E}$ such that $0 < \mathfrak{E}(\omega) < \infty$ for all $\omega \in \widetilde{\Omega_1}(SG)$.

Proof.

100% bare hands. □

- for vector equations **both** approaches matter
- their exact relation is not yet understood.

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