Projections of measures invariant under the geodesic flow

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Introduction

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- Mattila (1975): higher dimensional case

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- Kaufman (1968), Mattila (1990), Hu and Taylor (1994), Falconer and Mattila (1996)

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- Sauer and York (1997), Hunt and Kaloshin (1997): smooth mappings
- Peres and Schlag (2000): Sobolev dimensions of measures on compact metric spaces and parametrized families of transversal mappings

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Assume that $\varphi_*\mu = \mu$.

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Theorem (Ledrappier and Lindenstrauss 2003)

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Answers (J^2 and Leikas 2005)

- If $\iint |x-y|^{-\alpha}\, d\mu(x) d\mu(y) < \infty$ for $\alpha > 2$, then $\Pi_* \mu$ has fractional derivatives of order γ in L^2 for all $\gamma < (\alpha - 2)/2$.

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Tools

- Peres-Schlag formalism

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Answer (Hovila, J^2 and Ledrappier 2011)

Not necessarily.

Motivation: Quantum Unique Ergodicity

- Let ψ_n be a sequence of orthonormal eigenfunctions of the Laplacian on M.

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- Anantharaman and Nonnenmacher (2007): On general hyperbolic surfaces any limit is the projection of a φ-invariant measure with dimension at least 2.
- Rivière (2010): Surfaces with variable negative curvature have the same property.

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Theorem (Hovila, J^2 and Ledrappier 2011)

Let M be a compact surface with variable negative curvature. There exists an ergodic φ-invariant measure μ on T^1M such that $\dim \Pi_* \mu = 2$ and $\Pi_* \mu \perp \mathcal{L}^2$.

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Remark

It suffices to construct an ergodic φ-invariant measure μ such that $\dim \mu = 2$ and $\mu \perp \mathcal{H}^2$.

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- Ratner (1969): There exists a symbolic coding of a hyperbolic flow.
- Consider Markov measures on the symbolic coding of the geodesic flow.

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- Let σ be the left shift on Σ.
- Let r be a positive function on σ.
- Define the special flow $(\widetilde{\Sigma}_r, \tilde{\sigma}_t)$ by translation on the second coordinate where

$$\widetilde{\Sigma}_r = \{(\underline{\omega}, s) \mid \underline{\omega} \in \Sigma, 0 \leq s \leq r(\underline{\omega})\}/(\underline{\omega}, r(\underline{\omega})) \sim (\sigma(\underline{\omega}), 0).$$

A sketch of the proof: dimension

- Ratner (1969): There is a mixing subshift of finite type (Σ, σ), Hölder functions $r : \Sigma \to \mathbb{R}$ and $\pi : \widetilde{\Sigma}_r \to T^1M$ such that the following diagram is commutative

$$\begin{array}{ccc} \widetilde{\Sigma}_r & \xrightarrow{\tilde{\sigma}_t} & \widetilde{\Sigma}_r \\ \downarrow \pi & & \downarrow \pi \\ T^1M & \xrightarrow{\varphi_t} & T^1M \end{array}$$

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- Let $P = P_{ij}$ be a Markov matrix such that $P_{ij} > 0$ if and only if $A_{ij} = 1$.

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- Let $P = P_{ij}$ be a Markov matrix such that $P_{ij} > 0$ if and only if $A_{ij} = 1$.
- Let μ_P be a σ-invariant Markov measure on Σ.
- Define a $\widetilde{\sigma}$-invariant probability measure $\widetilde{\mu}_P$ on $\widetilde{\Sigma}_r$ by

$$\widetilde{\mu}_P = \frac{\int_\Sigma \mathcal{L}|_{[0,r)}\, d\mu_P}{\int_\Sigma r\, d\mu_P}.$$

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- This is guaranteed by using l-step Markov measures.

A sketch of the proof: singularity

- The aim is to show that P can be chosen in such a way that

$$\limsup_{\varepsilon \to 0} \frac{m_P(B((x,v),\varepsilon))}{\varepsilon^2} = \infty$$

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- Define observables (X_n^u, Y_n^u) and (X_n^s, Y_n^s) such that X_n describes the mass and Y_n describes the radius.

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- Define observables (X_n^u, Y_n^u) and (X_n^s, Y_n^s) such that X_n describes the mass and Y_n describes the radius.
- The vector valued almost sure invariance principle (Melbourne and Nicol 2009) implies that (X_n^u, Y_n^u) and (X_n^s, Y_n^s) can be approximated by 2-dimensional Brownian motions.

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- This means that for $v \in \{u, s\}$ there exist $\lambda > 0$ and a probability space $(X, \mathbb{P})$ supporting a sequence of random variables $(\widetilde{X}_n^v, \widetilde{Y}_n^v)$ having the same distribution as (X_n^v, Y_n^v) and a 2-dimensional Brownian motion W^v with a covariance matrix Q^v such that

$$|(\widetilde{X}_n^v, \widetilde{Y}_n^v) - W^v(n)| \ll n^{\frac{1}{2}-\lambda}$$

  $\mathbb{P}$-almost surely for large n.

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- The fluctuations of the observables are large enough.
- This implies the claim.

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Crucial:

- To show that the covariance matrix is nonsingular.

A sketch of the proof: singularity

Crucial:

- To show that the covariance matrix is nonsingular.
- To handle stable and unstable manifolds simultaneously.

Remark

On a certain class of Riemann surfaces with constant negative curvature and with boundary, there exist a φ-invariant measure μ such that $\dim \mu = \dim(\operatorname{spt} \mu) = \dim(\Pi_* \mu) = 2$ and $\mathcal{L}^2(\operatorname{spt} \Pi_* \mu) = 0$.

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- Main tool: Besicovitch-Federer projection theorem for transversal families of mappings

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- Main tool: Besicovitch-Federer projection theorem for transversal families of mappings

Theorem (Hovila, J^2 and Ledrappier 2011)

Let $E \subset \mathbb{R}^n$ be $\mathcal{H}^m$-measurable with $\mathcal{H}^m(E) < \infty$. Assume that $\Lambda \subset \mathbb{R}^l$ is open and $\{P_\lambda : \mathbb{R}^n \to \mathbb{R}^m\}_{\lambda \in \Lambda}$ is a transversal family of maps. Then E is purely m-unrectifiable, if and only if $\mathcal{H}^m(P_\lambda(E)) = 0$ for $\mathcal{L}^l$-almost all $\lambda \in \Lambda$.