Singularity Spectra of Signed Mandelbrot Cascades

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Outline

Definition

$(\Omega, \mathcal{F}, \mathbb{P})$: a probability space.

$W \geq 0$: a nonnegative random variable with $\mathbb{E}(W) = 1$.

Take a sequence of independent copies of W, encoded by dyadic words:

$$\{W(a) : a \in \cup_{n\geq 1}\{0,1\}^n\}.$$

For example

$$W(0),\ W(01),\ W(010),\ W(0100), \cdots$$

are i.i.d. random variables with expectation 1.

Definition

For $x \in [0,1]$ and $n \geq 1$ let

$$x|_n = x_1 x_2 \cdots x_n \in \{0,1\}^n$$

be the first n letters of the dyadic expansion of x.

For $n \geq 1$ define the random measure μ_n on $[0,1]$:

$$\mu_n(dx) = W(x_1)W(x_1x_2)\cdots W(x_1\cdots x_n)\,dx,$$

or equivalently define

$$F_n(x) = \int_0^x W(t_1)W(t_1t_2)\cdots W(t_1\cdots t_n)\,dt.$$

Definition

$\{\mu_n\}_{n\geq 1}$ is a measure-valued martingale:

$$\mathbb{E}(\mu_n(dx) \,|\, \mathcal{F}_m) = \mu_m(dx), \ \forall\ m \leq n,$$

where $\mathcal{F}_m = \sigma(W(a) : |a| \leq m)$.

Denote by μ the weak-limit of $\{\mu_n\}_{n\geq 1}$. We call μ the Mandelbrot cascade measure (generated by W).

A "toy" model of multiplicative chaos (Kolmogorov 41', Yaglom 66', Mandelbrot 70s')

Theorem of Kahane and Peyrière

Conjectured by Mandelbrot in 1974:

Theorem (Kahane & Peyrière 76)

If $\mathbb{E}(W\log_2 W) < 1$, then

- $\{\mu_n([0,1])\}_{n\geq 1}$ converges to μ in L^p for some $p > 1$;
- for any $p > 1$, $\mathbb{E}(\mu([0,1])^p) < \infty \Leftrightarrow \mathbb{E}(W^p) < 2^{p-1}$;
- almost surely for μ-almost every $x \in [0,1]$,

$$\lim_{n\to\infty} \frac{\log \mu(I_n(x))}{\log |I_n(x)|} = 1 - \mathbb{E}(W\log_2 W) > 0.$$

Example: $W = \exp(\gamma\mathcal{N}(0,1) - \gamma^2/2)$

$\mathbb{E}(W \log_2 W) < 1 \Leftrightarrow \gamma^2/\log 4 < 1.$

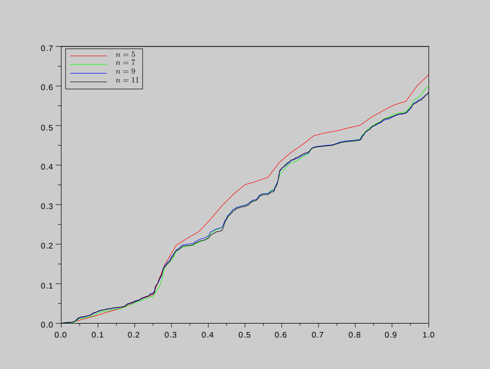

Figure: $F_n(x) = \int_0^x W(t_1)W(t_1t_2)\cdots W(t_1\cdots t_n)\,dt$ for $\gamma^2/\log 4 = 1/9$.

Erase the restriction of positivity

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Define the sequence of random piecewise linear functions F_n

For $x \in [0,1]$ and $n \geq 1$ let

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be the first n letters of the dyadic expansion of x.

For $n \geq 1$ define

$$F_n(x) = \int_0^x W(t_1)W(t_1 t_2)\cdots W(t_1 \cdots t_n)\, dt.$$

Fractional multiplicative processes

For $H \leq 1$ let

$$X_H = \begin{cases} +2^{1-H}, & \text{with probability } \frac{1+2^{-1+H}}{2}; \\ -2^{1-H}, & \text{with probability } \frac{1-2^{-1+H}}{2}. \end{cases}$$

Fractional multiplicative processes

Theorem (Barral & Mandelbrot 09)

If $H \in (1/2, 1]$ and $W = X_H$,

- $\{F_n\}_{n\geq 1}$ converges to a limit F almost surely and in L^p for all $p \geq 1$;
- almost surely for all $x \in [0,1]$,

$$\liminf_{y\to x, y\neq x} \frac{\log|F(y)-F(x)|}{\log|y-x|} = H;$$

- almost surely $\dim_H \mathrm{Graph}(F) = 2 - H$.

Fractional multiplicative processes

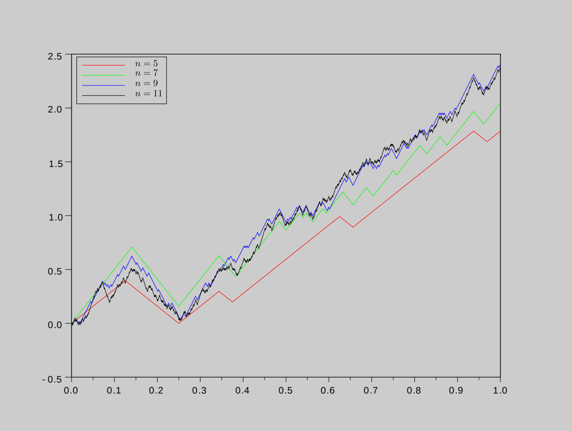


Figure: F_n for $H = 2/3$.

Signed Mandelbrot cascades

Let

$$\tau(q) = -1 + q - \log_2 \mathbb{E}(|W|^q).$$

Theorem (Barral, J. & Mandelbrot 10)

If there exists $q \in (1,2]$ such that $\tau(q) > 0$,

- $\{F_n\}_{n\geq 1}$ converges uniformly to a limit F almost surely and in L^p for some $p > 1$;
- almost surely F is α-Hölder continuous for any $\alpha \in (0, \max_{p>1} \tau(p)/p)$.

Example: $W = X_H \cdot \exp(\gamma\mathcal{N}(0,1) - \gamma^2/2)$

Suppose X_H and $\mathcal{N}(0,1)$ are independent. We have

$$\begin{aligned}\tau(q) &= -1 + q - \log_2 \mathbb{E}(|W|^q) \\ &= -1 + \left(H + \frac{\gamma^2}{\log 4}\right) q - \frac{\gamma^2}{\log 4} q^2.\end{aligned}$$

The existence of $q \in (1,2]$ such that $\tau(q) > 0$ is equivalent to

$$\frac{\gamma^2}{\log 4} < 1 \text{ and } \begin{cases} 2\sqrt{\frac{\gamma^2}{\log 4}} - \frac{\gamma^2}{\log 4} < H \leq 1, & \text{if } \frac{\gamma^2}{\log 4} \geq \frac{1}{4}; \\ \frac{1}{2} + \frac{\gamma^2}{\log 4} < H \leq 1, & \text{if } \frac{\gamma^2}{\log 4} < \frac{1}{4}. \end{cases}$$

Example: $W = X_H \cdot \exp(\gamma\mathcal{N}(0,1) - \gamma^2/2)$

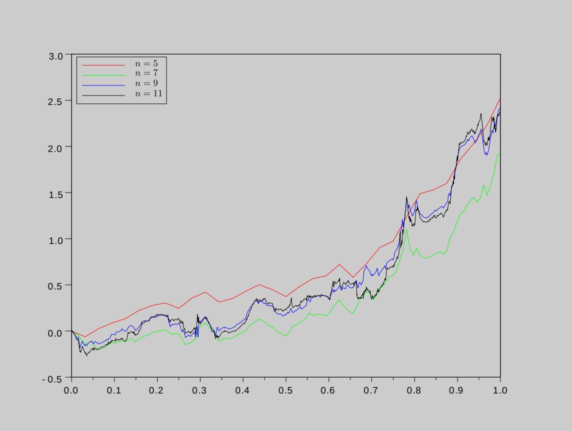
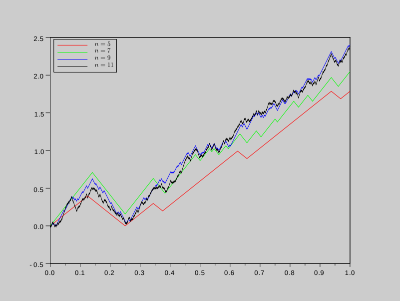
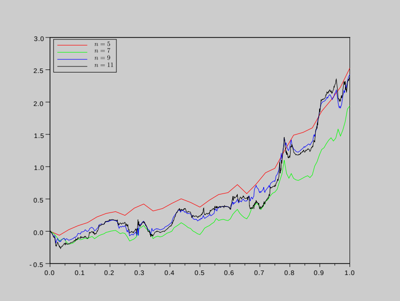


Figure: F_n for $\frac{\gamma^2}{\log 4} = \frac{1}{9}$ and $H = \frac{2}{3}$.

Comparing


n = 5
n = 7
n = 9
n = 11
2.5
2.0
1.5
1.0
0.5
0.0
-0.5
0.0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0


n = 5
n = 7
n = 9
n = 11
3.0
2.5
2.0
1.5
1.0
0.5
0.0
-0.5
0.0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0

Multifractal analysis of functions

For $x \in [0,1]$ define

$$h_F(x) = \liminf_{r\to 0} \frac{1}{\log r} \log O_F(B(x,r)),$$

where given $B \subset [0,1]$, $O_F(B) = \sup_{s,t\in B} |F(s) - F(t)|$ is the oscillation of F over B. For $\alpha \geq 0$ define

$$E_F(\alpha) = \{x \in [0,1] : h_F(x) = \alpha\}.$$

The singularity spectrum of F is the function

$$d_F : \alpha \geq 0 \mapsto \dim_H E_F(\alpha).$$

Multifractal analysis of functions

For $q \in \mathbb{R}$ let

$$\tau_F(q) = \liminf_{r\to 0} \frac{1}{\log r} \log \sup_{\mathcal{B}_r} \sum_{B\in\mathcal{B}_r} O_F(B)^q,$$

where the supreme is taken over all possible r-packing of the set $\{x \in [0,1] : O_F(B(x,r)) > 0 \text{ for all } r > 0\}$.

For any $\alpha \geq 0$

$$\dim_H E_F(\alpha) \leq \tau_F^*(\alpha) := \inf_{q\in\mathbb{R}} q\alpha - \tau_F(q).$$

Singularity spectrum of signed Mandelbrot cascade

Recall that $\tau(q) = -1 + q - \log_2 \mathbb{E}(|W|^q)$. Suppose that $\mathbb{E}(|W|^q) < \infty$ for all $q \in \mathbb{R}$ and denote by

$$J = \{q : q\tau'(q) - \tau(q) > 0\}.$$

Theorem (Barral & J. 10)

With probability one: (1)

$$\tau_F(q) = \begin{cases} \tau(q), & \text{if } q \in J; \\ \tau'(\overline{q}) \cdot q, & \text{if } \overline{q} = \sup J < \infty \text{ and } q \in [\overline{q}, \infty); \\ \tau'(\underline{q}) \cdot q, & \text{if } \underline{q} = \inf J > -\infty \text{ and } q \in (-\infty, \underline{q}]; \end{cases}$$

(2) for any $\alpha \geq 0$ we have

$$\dim_H E_F(\alpha) = \tau_F^*(\alpha).$$

Example: $W = X_H \cdot \exp(\gamma \mathcal{N}(0,1) - \gamma^2/2)$

$$\tau(q) = -1 + \left(H + \frac{\gamma^2}{\log 4}\right) q - \frac{\gamma^2}{\log 4} q^2.$$

Assumption:

$$\frac{\gamma^2}{\log 4} < 1 \text{ and } \begin{cases} 2\sqrt{\frac{\gamma^2}{\log 4}} - \frac{\gamma^2}{\log 4} < H \leq 1, & \text{if } \frac{\gamma^2}{\log 4} \geq \frac{1}{4}; \\ \frac{1}{2} + \frac{\gamma^2}{\log 4} < H \leq 1, & \text{if } \frac{\gamma^2}{\log 4} < \frac{1}{4}. \end{cases}$$

We have

$$J = \left(-\sqrt{\frac{\log 4}{\gamma^2}}, \sqrt{\frac{\log 4}{\gamma^2}}\right) \text{ and } \tau^*(\alpha) = 1 - \left(\frac{H + \frac{\gamma^2}{\log 4} - \alpha}{2\sqrt{\frac{\gamma^2}{\log 4}}}\right)^2.$$

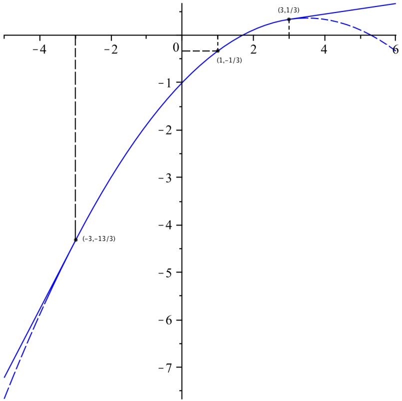


Figure: $\tau_F(q) = \begin{cases} -1 + (\frac{2}{3} + \frac{1}{9})q - \frac{1}{9}q^2, & \text{if } q \in J = (-3,3); \\ \frac{1}{9} \cdot q, & \text{if } q \in [3,\infty); \\ \frac{13}{9} \cdot q, & \text{if } q \in (-\infty,-3]. \end{cases}$

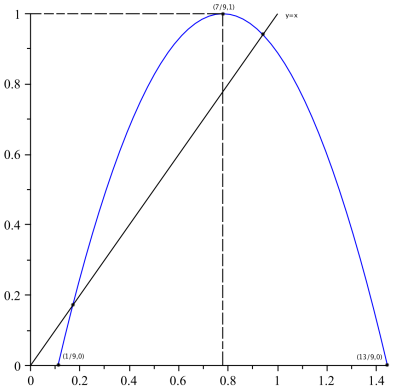


Figure: $\tau_F^*(\alpha) = \begin{cases} 1 - \left(\frac{\frac{2}{3}+\frac{1}{9}-\alpha}{\frac{2}{3}}\right)^2, & \text{if } \alpha \in \left[\frac{1}{9}, \frac{13}{9}\right]; \\ -\infty, & \text{elsewhere.} \end{cases}$

Hausdorff dimension of the graph

Theorem (J. 11)

Almost surely

$$\dim_H \mathrm{Graph}(F) = \dim_B \mathrm{Graph}(F) = 1 - \tau_F(1).$$

In the case of $W = X_H \cdot \exp(\gamma \mathcal{N}(0,1) - \gamma^2/2)$ and

$$\tau(q) = -1 + \left(H + \frac{\gamma^2}{\log 4}\right) q - \frac{\gamma^2}{\log 4} q^2,$$

we have

$$\dim_H \mathrm{Graph}(F) = \dim_B \mathrm{Graph}(F) = 2 - H.$$

Graph, range and level set singularity spectra

Recall $E_F(\alpha) = \{x \in [0,1] : h_F(x) = \alpha\}$. Define

$$\begin{aligned} G_F(\alpha) &= \{(x, F(x)) : x \in E_F(\alpha)\}\,; \\ R_F(\alpha) &= \{F(x) : x \in E_F(\alpha)\}\,; \\ L_F^y(\alpha) &= \{(x, y) : x \in E_F(\alpha)\}\,, \end{aligned}$$

for $y \in R_F(\alpha)$. We call the functions

$$\begin{aligned} d_F^G &: \alpha \geq 0 \mapsto \dim_H G_F(\alpha); \\ d_F^R &: \alpha \geq 0 \mapsto \dim_H R_F(\alpha); \\ d_F^{L^y} &: \alpha \geq 0 \mapsto \dim_H L_F^y(\alpha) \end{aligned}$$

the graph, range and level set singularity spectra of F.

Graph and range singularity spectra

Theorem (J. 11)

With probability one for all $\alpha \geq 0$,

$$d_F^G(\alpha) = \left(\frac{\tau_F^*(\alpha)}{\alpha} \wedge (1 - \alpha + \tau_F^*(\alpha))\right) \vee \tau_F^*(\alpha),$$

$$d_F^R(\alpha) = \frac{\tau_F^*(\alpha)}{\alpha} \wedge 1.$$

Graph and range singularity spectra

Theorem (J. 11)

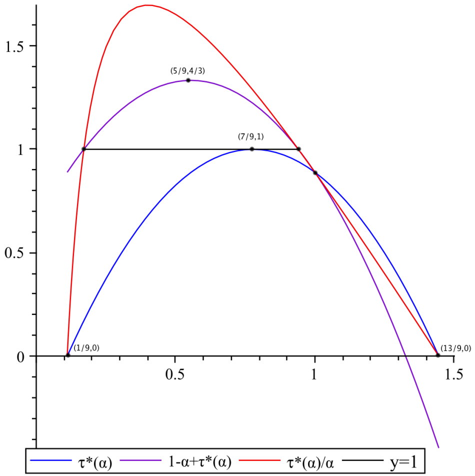
With probability one for all $\alpha \geq 0$,

$$d_F^G(\alpha) = \begin{cases} \frac{\tau_F^*(\alpha)}{\alpha}, & \text{if } \tau_F^*(\alpha) \leq \alpha \leq 1; \\ 1 - \alpha + \tau_F^*(\alpha), & \text{if } \alpha < \tau_F^*(\alpha); \\ \tau_F^*(\alpha), & \text{if } \alpha > 1. \end{cases}$$

$$d_F^R(\alpha) = \begin{cases} \frac{\tau_F^*(\alpha)}{\alpha}, & \text{if } \alpha \geq \tau_F^*(\alpha); \\ 1, & \text{if } \alpha < \tau_F^*(\alpha). \end{cases}$$

Remark: (1) $d_F^G(\tau_F'(1)) = 1 - \tau_F(1) = \dim_H \mathrm{Graph}(F)$;

(2) $d_F^G(\alpha) = d_F(\alpha) + d_F^R(\alpha) \cdot (1 - \alpha) \vee 0$.


1.5
1
0.5
0
0.5
1
1.5
(5/9,4/3)
(7/9,1)
(1/9,0)
(13/9,0)
τ*(α)
1-α+τ*(α)
τ*(α)/α
y=1

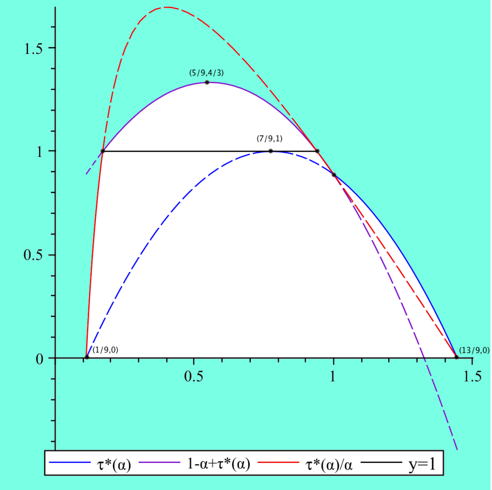


Figure: $\left(\frac{\tau_F^*(\alpha)}{\alpha} \wedge \left(\tau_F^*(\alpha) + 1 - \alpha\right)\right) \vee \tau_F^*(\alpha)$.

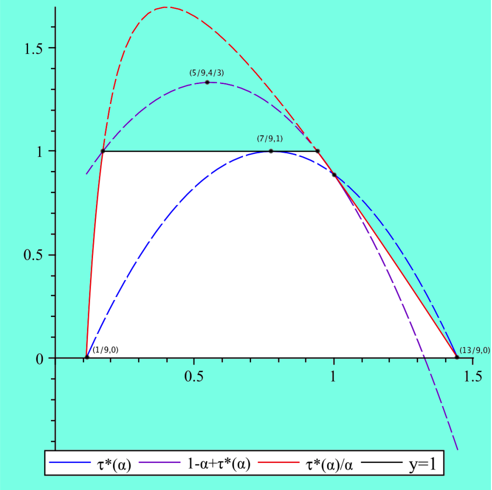


Figure: $\frac{\tau_F^*(\alpha)}{\alpha} \wedge 1$.

Level set singularity spectrum

Theorem (J. 11)

With probability one for all $\alpha \geq 0$,

$$d_F^G(\alpha) = \begin{cases} \frac{\tau_F^*(\alpha)}{\alpha}, & \text{if } \tau_F^*(\alpha) \leq \alpha \leq 1; \\ 1 - \alpha + \tau_F^*(\alpha), & \text{if } \alpha < \tau_F^*(\alpha); \\ \tau_F^*(\alpha), & \text{if } \alpha > 1. \end{cases}$$

$$d_F^R(\alpha) = \begin{cases} \frac{\tau_F^*(\alpha)}{\alpha}, & \text{if } \alpha \geq \tau_F^*(\alpha); \\ 1, & \text{if } \alpha < \tau_F^*(\alpha). \end{cases}$$

Level set singularity spectrum in Lebesgue almost every direction

Denote by l_θ the line in $\mathbb{R}^2$ passing through the origin and making an angle θ with the x-axis.

For any $y \in l_\theta$, denote by $l^\perp_{y,\theta}$ the line perpendicular to l_θ, passing through y.

Denote by $R_{F,\theta}(\alpha)$ the orthogonal projection of $G_F(\alpha)$ onto l_θ.

For each $y \in R_{F,\theta}(\alpha)$ we define the level set of F in θ-direction by

$$L^y_{F,\theta}(\alpha) = G_F(\alpha) \cap l^\perp_{y,\theta}.$$

Level set singularity spectrum in Lebesgue almost every direction

Theorem (J.)

Almost surely for Lebesgue almost every $\theta \in (0, \pi)$, for all $\alpha \in (0, 1)$ such that $\alpha < \tau_F^*(\alpha)$,

- there exists a positive measure $\mu_{\alpha,\theta}^R$ carried by $R_{F,\theta}(\alpha)$ which is absolutely continuous w.r.t. Lebesgue measure;
- for $\mu_{\alpha,\theta}^R$ almost every $y \in R_{F,\theta}(\alpha)$ we have

$$\dim_H L_{F,\theta}^y(\alpha) = d_F^G(\alpha) - 1 = \tau_F^*(\alpha) - \alpha.$$

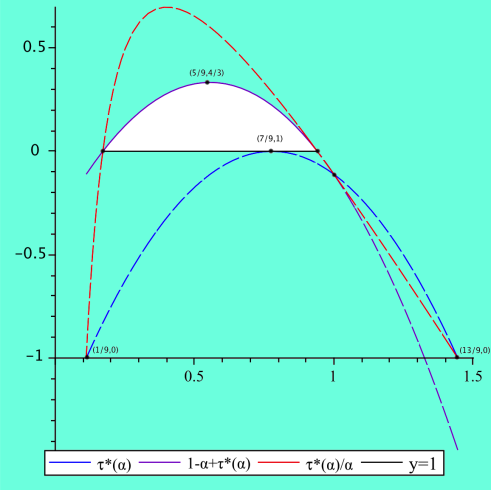


Figure: $\tau_F^*(\alpha) - \alpha$.

Thanks!