Local homogeneity and dimensions of measures

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Porquerolles, 13th June 2011

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This talk mainly exhibits a recent work with
Tapio Rajala (Pisa) and Ville Suomala (Oulu).

Local homogeneity and dimensions of measures

1. **Upper conical density results**

2. **Dimension estimates for porous measures**
   - Large porosity
   - Small porosity

3. **Local multifractal analysis**

Conical densities

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- The main applications deal with rectifiability and porosity.
- The study of conical densities go back to Besicovitch (1938), Morse and Randolph (1944), Marstrand (1954), Federer (1969), Salli (1985), and Mattila (1988).
- Recent work include Suomala and K. (2008), Csörnyei, Rajala, Suomala, and K. (2010), Suomala and K. (2011), and Rajala, Suomala, and K. (preprint).

Upper density result for Hausdorff measures

Theorem (Besicovitch 1938 and Marstrand 1954)

Suppose $0 \le s \le n$ and $A \subset \mathbb{R}^n$ with $0 < \mathcal{H}^s(A) < \infty$. Then

$$2^{-s} \le \limsup_{r \downarrow 0} \frac{\mathcal{H}^s\big(A \cap B(x,r)\big)}{(2r)^s} \le 1$$

for $\mathcal{H}^s$-almost all $x \in A$.

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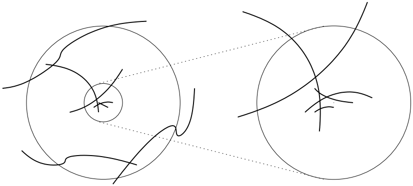

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Upper conical densities

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- If we know that the measure is "scattered enough", can we say how the measure is distributed on those scales where we have a lot of mass?
- If the measure is purely unrectifiable and doubling, then the answer is yes. An example of Csörnyei, Rajala, Suomala, and K. (2010) shows that it is really needed that the measure is doubling.

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- If the measure is purely unrectifiable and doubling, then the answer is yes. An example of Csörnyei, Rajala, Suomala, and K. (2010) shows that it is really needed that the measure is doubling.
- Another possibility is to assume that the dimension of the measure is large enough.

Definition of nonsymmetric cones

Let $G(n, n-m)$ denote the space of all $(n-m)$-dimensional linear subspaces of $\mathbb{R}^n$ and set $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$.

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For $x \in \mathbb{R}^n$, $r > 0$, $V \in G(n, n-m)$, $\theta \in S^{n-1}$, and $0 < \alpha \le 1$ define

$$X(x, r, V, \alpha) = \{y \in B(x, r) : \mathrm{dist}(y - x, V) < \alpha |y - x|\},$$
$$H(x, \theta, \alpha) = \{y \in \mathbb{R}^n : (y - x) \cdot \theta > \alpha |y - x|\}.$$

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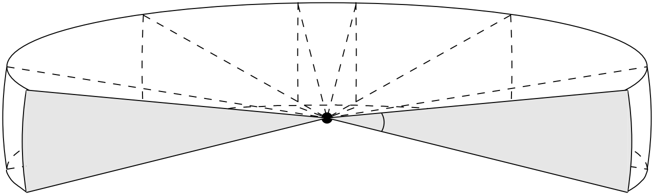

The set $X(x, r, V, \alpha) \setminus H(x, \theta, \alpha)$ when $n = 3$ and $m = 1$.

Upper conical density result for packing measures

Theorem (Suomala & K. 2008)

Suppose $0 \le m < s \le n$ and $0 < \alpha \le 1$. Then there exists $c = c(n,m,s,\alpha) > 0$ so that for every $A \subset \mathbb{R}^n$ with $0 < \mathcal{P}^s(A) < \infty$ we have

$$\limsup_{r \downarrow 0} \inf_{\substack{\theta \in S^{n-1} \\ V \in G(n,n-m)}} \frac{\mathcal{P}^s\big(A \cap X(x,r,V,\alpha) \setminus H(x,\theta,\alpha)\big)}{(2r)^s} \ge c$$

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for $\mathcal{P}^s$-almost all $x \in A$.

To our knowledge, this is the first upper conical density result for other measures than the Hausdorff measures.

Upper conical density result for general measures

Theorem (Rajala & Suomala & K. preprint)

Suppose $0 \le m < s \le n$ and $0 < \alpha \le 1$. Then there exists $c = c(n, m, s, \alpha) > 0$ so that for every Radon measure μ on $\mathbb{R}^n$ with $\underline{\dim}_{\mathrm{p}}(\mu) \ge s$ we have

$$\limsup_{r \downarrow 0} \inf_{\substack{\theta \in S^{n-1} \\ V \in G(n, n-m)}} \frac{\mu\big(X(x, r, V, \alpha) \setminus H(x, \theta, \alpha)\big)}{\mu\big(B(x, r)\big)} > c$$

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The proof of this result uses a local homogeneity estimate.

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- Porosity was introduced by Denjoy (1920). His definition is nowadays called *upper porosity*. Although upper porosity is useful in many connections, one cannot get nontrivial dimension estimates for upper porous sets.
- Dimension estimates obtained from *lower porosity* were used by Sarvas (1975), Trocenko (1981), and Väisälä (1987) in connection with the boundary behavior of quasiconformal mappings.
- In lower porosity we have holes on all scales whereas in upper porosity we just know that there are arbitrary small scales having holes.

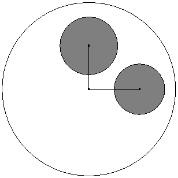
Porosity of sets

Let $A \subset \mathbb{R}^n$, $k \in \{1, \ldots, d\}$, $x \in A$, and $r > 0$. We define

$$\begin{aligned} \mathrm{por}_k(A, x, r) = \sup\{\varrho \geq 0 : {} & \text{there are } y_1, \ldots, y_k \in \mathbb{R}^n \text{ such that} \\ & B(y_i, \varrho r) \subset B(x, r) \setminus A \text{ for every } i \\ & \text{and } (y_i - x) \cdot (y_j - x) = 0 \text{ if } i \neq j\} \end{aligned}$$

and from this the (lower) *k-porosity of A at x* as

$$\mathrm{por}_k(A, x) = \liminf_{r \downarrow 0} \mathrm{por}_k(A, x, r).$$



Recent results

- For recent results on the dimension of porous sets, see Järvenpää, Järvenpää, Suomala, and K. (2005), Rajala (2009), Chousionis (2009), Järvenpää, Järvenpää, Rajala, Rogovin, Suomala, K. (2010), and Suomala and K. (2011).

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- We also define porosity for measures. In applications, it is more convenient to consider measures instead of sets.

- For recent results concerning porous measures, see Suomala and K. (2008), Beliaev, Järvenpää, Järvenpää, Rajala, Smirnov, Suomala, K. (2009), Rajala, Suomala, K. (preprint), and Shmerkin (2011).

Porosity of measures

Let μ be a Radon measure on $\mathbb{R}^n$, $k \in \{1, \ldots, d\}$, $x \in \mathbb{R}^n$, $r > 0$, and $\varepsilon > 0$. We set

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and the *k-porosity of the measure* μ *at* x is defined to be

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$$\mathrm{por}_k(\mu, x) = \lim_{\varepsilon \downarrow 0} \liminf_{r \downarrow 0} \mathrm{por}_k(\mu, x, r, \varepsilon).$$

An example of Smirnov et al. (2009) shows that even if $\mathrm{por}_1(\mu, x) > 0$ in a set of positive μ-measure, it is possible that $\mu(A) = 0$ for all $A \subset \mathbb{R}^n$ with $\inf_{x \in A} \mathrm{por}_1(A, x) > 0$.

Dimension estimate (when the porosity is large)

Theorem (Rajala & Suomala & K. preprint)

There exists a constant $c > 0$ such that for every Radon measure μ on $\mathbb{R}^n$ we have

$$\overline{\dim}_{\text{loc}}(\mu, x) \leq n - k + \frac{c}{-\log\big(1 - 2\operatorname{por}_k(\mu, x)\big)}$$

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The proof of this result uses a local homogeneity estimate.

Dimension estimate (when the porosity is small)

Theorem (Rajala & Suomala & K. preprint)

There exists a constant $c > 0$ such that for every Radon measure μ on an s-regular metric space X satisfying the density point property we have

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The proof of this result uses a local homogeneity estimate.

Observe that this is an application of the local homogeneity in metric spaces.

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- By using δ-partitions, we are able to overcome the technical problems ordinarily caused by the interplay between cubes and balls.

- In doubling metric spaces, it is possible to define "dyadic cubes" i.e. δ-partitions that are nested.

Moran constructions

Let X be a complete doubling metric space, $\{E_{\mathtt{i}} : \mathtt{i} \in \Sigma_*\}$ a Moran construction on X satisfying the strong separation condition and E its limit set. The canonical projection of $\mathtt{i} \in \Sigma = \{1, \ldots, m\}^{\mathbb{N}}$ is denoted by $x_{\mathtt{i}} \in E$.

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We assume that there is a continuous function $r_i \colon E \to (0,1)$ for all $i \in \{1, \ldots, m\}$. Let $r_{\mathtt{i}}(x) = \prod_{k=1}^{n} r_{i_k}(x)$ and assume that

$$\lim_{n\to\infty} \frac{\log \operatorname{diam}(E_{\mathtt{i}|_n})}{\log r_{\mathtt{i}|_n}(x_{\mathtt{i}})} = 1$$

uniformly for all $\mathtt{i} \in \Sigma$.

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A probability measure μ induces probability vectors $p_{\mathtt{i}} = (p_{\mathtt{i}}^1, \ldots, p_{\mathtt{i}}^m)$ for which $\mu(E_{\mathtt{i}i}) = p_{\mathtt{i}}^i \mu(E_{\mathtt{i}})$.

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We assume that the weights $p_{\mathtt{i}}$ are controlled in terms of a continuous probability function $p(x) = \big(p_1(x), \ldots, p_m(x)\big)$.

Moran constructions

Theorem (Rajala & Suomala & K. preprint)

If $p_{\mathtt{i}|_n} \to p(x_{\mathtt{i}})$ as $n \to \infty$ uniformly for all $\mathtt{i} \in \Sigma$, then, for all $x \in E$ and all $q \geq 0$, $\tau_q(\mu, x)$ is the unique $\tau \in \mathbb{R}$ that satisfies

$$\sum_{i=1}^{m} p_i(x)^q r_i(x)^{-\tau} = 1.$$

Moreover,

$$\dim_1(\mu, x) = \dim_{\mathrm{loc}}(\mu, x) = \frac{\sum_{i=1}^{m} p_i(x) \log p_i(x)}{\sum_{i=1}^{m} p_i(x) \log r_i(x)}$$

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Assuming a bit more on the Moran construction, we are able to show that μ satisfies the local multifractal formalism.

Local multifractal analysis

Let $E(\mu, \alpha, \varepsilon) = \{x \in X : \alpha - \varepsilon \le \underline{\dim}_{\mathrm{loc}}(\mu, x) \le \overline{\dim}_{\mathrm{loc}}(\mu, x) \le \alpha + \varepsilon\}$ and define the local Hausdorff and packing multifractal spectra of μ by setting

$$f_{\mathrm{H}}(\mu, \alpha, x) = \lim_{\varepsilon \downarrow 0} \lim_{r \downarrow 0} \dim_{\mathrm{H}}\bigl(E(\mu, \alpha, \varepsilon) \cap B(x, r)\bigr),$$

$$f_{\mathrm{p}}(\mu, \alpha, x) = \lim_{\varepsilon \downarrow 0} \lim_{r \downarrow 0} \dim_{\mathrm{p}}\bigl(E(\mu, \alpha, \varepsilon) \cap B(x, r)\bigr).$$

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Theorem (Rajala & Suomala & K. preprint)

If $p_{\mathtt{i}|_n} \to p(x_{\mathtt{i}})$ as $n \to \infty$ uniformly for all $\mathtt{i} \in \Sigma$, then

$$f_{\text{H}}(\mu,\alpha,x) = f_{\text{p}}(\mu,\alpha,x) = \inf_{q \in \mathbb{R}} \{\alpha q - \tau_q(\mu,x)\}$$

for all $x \in E$ and $\alpha_{\min}(x) \le \alpha \le \alpha_{\max}(x)$.

Here $\alpha_{\min}(x)$ and $\alpha_{\max}(x)$ are the asymptotic derivatives of $q \mapsto \tau_q(\mu,x)$.

Thank you!

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