Dimension gaps for Bernoulli approximations of conformal IFS

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(joint work with Mariusz Urbański and Bernd O. Stratmann)

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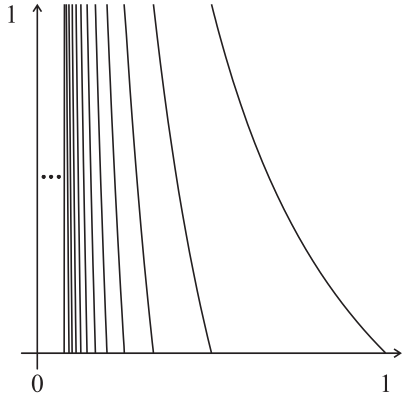
Fractals and related fields II

- The *Gauss map* $\mathfrak{G} : [0,1] \setminus \mathbb{Q} \to [0,1] \setminus \mathbb{Q}$ is given by

$$\mathfrak{G}(x) := \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor .$$



- $\sigma : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ shift dynamical systems over the full shift (with countable alphabet).
- $\omega \in \mathbb{N}^n$ we denote by $[\omega] := \{\tau \in \Sigma : \tau_{|n} = \omega\}$ the *cylinder set* of ω.
- $\pi : \mathbb{N}^{\mathbb{N}} \to (0,1) \setminus \mathbb{Q}$ canonical coding via continued fraction, that is
$$\pi(a_1, a_2, \dots) := \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}}.$$
- The Gauss map is conjugated to the left shift, i.e. $\pi \circ \sigma = \mathfrak{G} \circ \pi$.

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- $\mathscr{P} := \{p := (p_1, p_2, \ldots) : p_i \geq 0, \sum p_i = 1\}$, set of probability vectors.
- For $p \in \mathscr{P}$ we let m_p denote the *p-Bernoulli measure* on $\mathbb{N}^{\mathbb{N}}$ and by $\mu_p := m_p \circ \pi^{-1}$ its push forward onto $[0,1]$.
- Let $\varphi := \log(|\mathfrak{G}' \circ \pi|)$ denote the *geometric potential.*
- Let $f_p := \sum (\log(p_i)) \mathbb{1}_{[i]}$ be a *Bernoulli potential*.

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- For a Borel measure μ on $[0,1]$ we define its Hausdorff dimension by

$$\mathrm{HD}(\mu) := \inf\{\dim_H(A) : A \in \mathscr{B}, \mu(A) = 1\}.$$

Theorem (Kifer, Peres, Weiss)

$$\sup_{p \in \mathscr{P}} \mathrm{HD}(\mu_p) < 1 - 10^{-7}.$$

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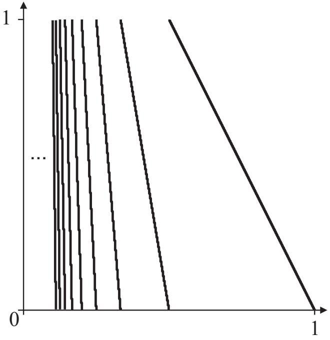
- For the so-called *alternating Lüroth map*[1], which is a linearised version of the Gauss map an analogue inequality does **not** hold.
- $L(x) := (1/n - x)(n(n+1))$, $x \in (1/(n+1), 1/n]$, $n \in \mathbb{N}$



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Observation 1: Linearised version

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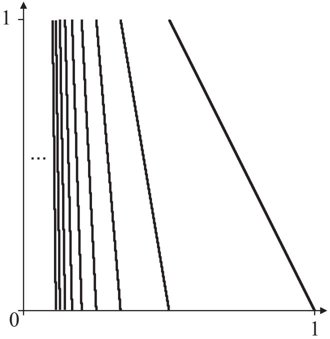

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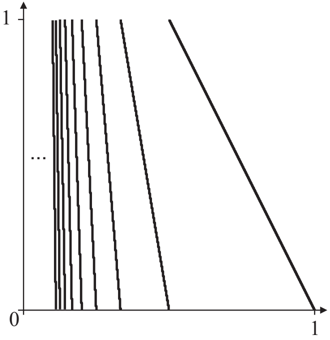

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- Expanding map with $k \geq 2$ full expanding diff'able branches $S : [0,1] \to [0,1]$, such that the geometric Potential $\varphi := \log|S'|$is Hölder.
- Consider the full shift $\Sigma_k := \{1,\ldots,k\}^{\mathbb{N}}$, the canonical coding map π, $\mathscr{M}$ the set of σ-invariant probability measures.
- Volume Lemma: (e.g. [Mauldin/Urbański '00] For $m \in \mathscr{M}$ we have

$$\mathrm{HD}\left(m \circ \pi^{-1}\right) = \frac{h_m(\sigma)}{\int \varphi \, dm}$$

- $\mathscr{M} \ni \mu \mapsto \dfrac{h_\mu(\sigma)}{\int \varphi \, d\mu}$ is upper semi-continuous.
- $\{m_p : (p_n) \in \mathscr{P}\}$ is a compact subset of $\mathscr{M}$.
- The unique measure of maximal dimension is the Gibbs measure for the potential φ , hence, if S is not essentially affine $\mathrm{HD}(\mu_p) < 1$ for all $(p_n) \in \mathscr{P}$.
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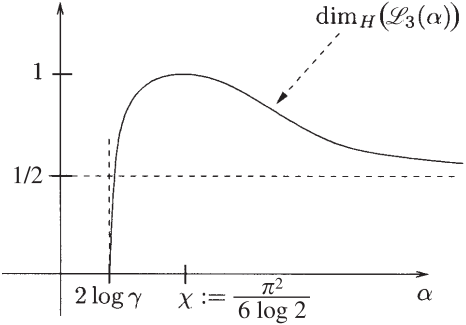
- Lyapunov spectrum for the Gauss map (K./Stratmann '07)

$$\mathscr{L}_3(\alpha) := \left\{ x \in [0,1] : \lim_{n\to\infty} S_n\varphi(x)/n = s \right\}.$$



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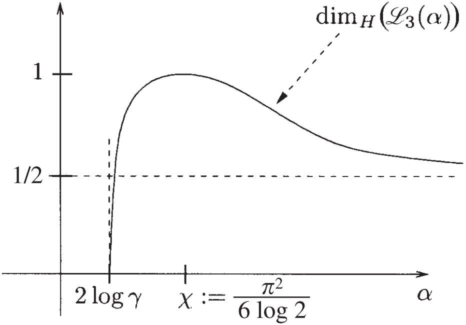

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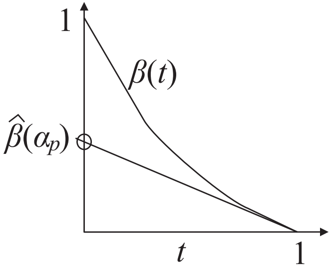
  $P(g) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{C\in\mathcal{C}_n} \exp\left(\sup_{x\in C} S_n g(x)\right)$
- Free energy function is defined for all $t \in \mathbb{R}$ by

$$\beta(t) := \inf\{b \in \mathbb{R} : P(b\varphi + tf_p) \le 0\}.$$



- [K./Jaerisch '10] For all $s \in \mathbb{R}$ we have

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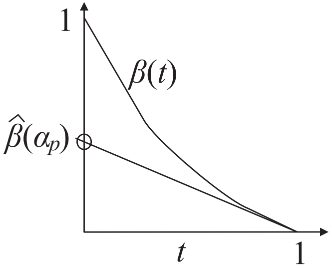

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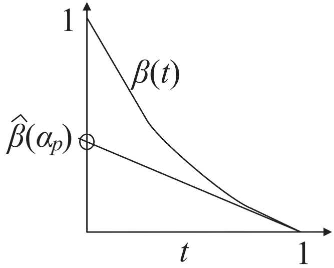

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- Hence, $t \in [0,1]$: $\beta(t) \ge (t-1)\alpha_p \implies \alpha_p \in \partial\beta(1)$.

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- The restriction of f_p and φ to Σ_k will be denoted by F_k and G_k, $k \in \mathbb{N}$. For $k \in \mathbb{N}$ define the real-analytic functions $\beta_k : \mathbb{R} \to \mathbb{R}$ implicitly by

$$P\left(uF_k + \beta_k(u)\, G_k\right) = 0, \qquad u \in \mathbb{R}.$$

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Lemma

Let φ, ψ be two Hölder continuous potentials on Σ_k . Let μ_ψ denote that Gibbs measure associated to ψ and assume that $\mu_\psi(\varphi) = 0$. We also assume that there exists a periodic point $x \in \Sigma_k$ with period $r_x \geq 1$ such that

$$\frac{1}{r_x} S_{r_x}\varphi(x) =: c \neq 0.$$

Then there exists $n \in \mathbb{N}$ such that

$$\sigma^2_{\mu_\psi}(\varphi) \geq \frac{\mu_\psi(C_n(x))}{2} c^2 > 0.$$

- W. l. o. g. (by adding a co-boundary) we have $\mathscr{L}_\psi \mathbb{1} = \mathbb{1}$.
- $\left\|\mathscr{L}_\psi^k \varphi\right\|_\alpha \leq C\rho^k$ for some $\rho \in (0,1) \implies F := \sum_{k=1}^\infty \mathscr{L}_\psi^k \varphi$ defines a Hölder function.
- $\widetilde{\varphi} := \varphi + F - F \circ \sigma \implies$

$$\begin{aligned}
\mathscr{L}_\psi(\widetilde{\varphi}) &= \mathscr{L}_\psi(\varphi) + \mathscr{L}_\psi(F) - \mathscr{L}_\psi(F \circ \sigma) \\
&= \mathscr{L}_\psi(\varphi) + \sum_{k=2}^\infty \mathscr{L}_\psi^k(\varphi) - \sum_{k=2}^\infty \mathscr{L}_\psi^k(\varphi \circ \sigma) \\
&= \sum_{k=1}^\infty \mathscr{L}_\psi^k(\varphi) - \sum_{k=2}^\infty \mathscr{L}_\psi^k(\varphi \circ \sigma) \\
&= \sum_{k=1}^\infty \mathscr{L}_\psi^k(\varphi) - \sum_{k=2}^\infty \mathscr{L}_\psi^{k-1}\left(\mathscr{L}_\psi(\varphi \circ \sigma)\right) \\
&= \sum_{k=1}^\infty \mathscr{L}_\psi^k(\varphi) - \sum_{k=2}^\infty \mathscr{L}_\psi^{k-1}\left(\varphi \cdot \mathscr{L}_\psi(\{1\})\right) = 0.
\end{aligned}$$

Proof of Lemma I

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$$

- $\mathscr{L}_\psi(\widetilde{\varphi}) = 0 \implies$

$$\begin{aligned}
\sigma^2_{\mu_\psi}(\varphi) &= \int \widetilde{\varphi}^2 \, d\mu_\psi + 2\sum_{k=1}^{\infty} \int \widetilde{\varphi} \cdot \widetilde{\varphi} \circ \sigma^k \, d\mu_\psi \\
&= \int \widetilde{\varphi}^2 \, d\mu_\psi + 2\sum_{k=1}^{\infty} \int \mathscr{L}_\psi^k \left(\widetilde{\varphi} \cdot \widetilde{\varphi} \circ \sigma^k \right) d\mu_\psi \\
&= \int \widetilde{\varphi}^2 \, d\mu_\psi + 2\sum_{k=1}^{\infty} \int \widetilde{\varphi} \cdot \mathscr{L}_\psi^k(\widetilde{\varphi}) \, d\mu_\psi = \int \widetilde{\varphi}^2 \, d\mu_\psi .
\end{aligned}$$

- The lemma follows since

$$\frac{1}{r_x} S_{r_x} \varphi(x) = \frac{1}{r_x} S_{r_x} \widetilde{\varphi}(x) .$$

- $\mathscr{L}_\psi(\widetilde{\varphi}) = 0 \implies$

$$\begin{aligned} \sigma^2_{\mu_\psi}(\varphi) &= \int \widetilde{\varphi}^2 \, d\mu_\psi + 2\sum_{k=1}^{\infty} \int \widetilde{\varphi} \cdot \widetilde{\varphi} \circ \sigma^k \, d\mu_\psi \\ &= \int \widetilde{\varphi}^2 \, d\mu_\psi + 2\sum_{k=1}^{\infty} \int \mathscr{L}^k_\psi \left(\widetilde{\varphi} \cdot \widetilde{\varphi} \circ \sigma^k \right) d\mu_\psi \\ &= \int \widetilde{\varphi}^2 \, d\mu_\psi + 2\sum_{k=1}^{\infty} \int \widetilde{\varphi} \cdot \mathscr{L}^k_\psi(\widetilde{\varphi}) \, d\mu_\psi = \int \widetilde{\varphi}^2 \, d\mu_\psi. \end{aligned}$$

- The lemma follows since

$$\frac{1}{r_x} S_{r_x} \varphi(x) = \frac{1}{r_x} S_{r_x} \widetilde{\varphi}(x).$$

Global bound on convexity of the Free Energy

Proof of $\sup_{p\in\mathscr{P}} \mathrm{HD}(\mu_p) < 1$

- Set $\varphi_{u,k} := \beta_k'(u)\, F_k + G_k, \quad \psi_{u,k} := u G_k + \beta_k(u)\, F_k.$
- The above lemma applied to the periodic points $x_{n,\ell} := (\ell, n, \ell, n, \ldots)$, $n \in \mathbb{N}$, $\ell \in \{1,2,3\}$, gives uniformly in $k \in \mathbb{N}$ and u from a compact interval in $[0,1]$ that

$$\beta_k''(u) = \frac{\sigma^2_{\psi_{u,k}}(\varphi_{u,k})}{-\int g\, d\mu_{\psi_{u,k}}} \geq C_1 \sum_{n>3, \ell=1,2,3} \mu_{\psi_{u,k}}(C_2(x_{n,\ell}))\, c^2_{x_{n,\ell}}/2,$$

where $c_x := \beta'(u)\, \varphi(x) + f_p(x)$.
- $c_{x_{n,\ell}}$ cannot be small for each $\ell = 1,2,3$.

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  where $c_x := \beta'(u) \varphi(x) + f_p(x)$.
- $c_{x_{n,\ell}}$ cannot be small for each $\ell = 1,2,3$.

- What about k-step Markov measures?
- What is the drop rate of the gap size if we increase k?
- Can we give a bound of the gap size in terms of $\left\{ I_n'' \right\}$?

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