On spectral properties of large dilute Wigner random matrices

O. Khorunzhiy

University of Versailles - Saint-Quentin, France

We study the spectral norm (maximal eigenvalue $\lambda_{\max}$) of $n \times n$ random real symmetric matrices $H^{(n,\rho)}$ whose elements $H_{ij}^{(n,\rho)}$, $i \le j$ are given by jointly independent random variables, similarly to the well-known ensemble of Wigner real symmetric matrices.

The difference between $H^{(n,\rho)}$ and the Wigner ensemble is that $H_{ij}^{(n,\rho)}$ is equal to 0 with probability $1 - \rho/n$ (dilute version). The concentration parameter $\rho = \rho_n$ represents the average number of non-zero elements per row in $H^{(n,\rho)}$.

Our results show that in the asymptotic regime when $\rho_n = n^\alpha$, $n \to \infty$, the value $\alpha = 2/3$ is the critical one with respect to the asymptotic behavior of $\lambda_{\max}$.

I.1. Dilute Wigner random matrices

$$H_{ij}^{(n,\rho)} = \frac{1}{\sqrt{\rho}} a_{ij}\, b_{ij}^{(n,\rho)}, \ 1 \le i \le j \le n,$$

where $\{a_{ij}, i \le j\}$ are jointly independent r.v. with symmetric probability distribution and

$$b_{ij}^{(n,\rho)} = \begin{cases} 1, & \text{with probability } \rho/n \\ 0, & \text{with probability } 1 - \rho/n \end{cases}$$

independent r.v. also independent from a_{ij}.

i) If $\rho = n$, then the matrix

$$H_{ij}^{(n)} = \frac{1}{\sqrt{n}} a_{ij}$$

represents the Wigner ensemble of real symmetric random matrices;

ii) $1 \ll \rho_n \ll n$, *dilute* version of Wigner RM;

iii) $\rho_n = O(1), n \to \infty$, *sparse* RM.

I.2. Semi-circle law (Wigner law)

a) Normalized eigenvalue counting function (NCF)

$$\sigma_n(\lambda) = \frac{1}{n}\,\#\left\{j : \lambda_j^{(n)} \le \lambda\right\}$$

converges as $n \to \infty$ to $\sigma_W(\lambda)$ with the density

$$\frac{d}{d\lambda}\sigma_W(\lambda) = \frac{1}{2\pi v^2}\sqrt{4v^2 - \lambda^2}, \quad |\lambda| \le 2v,$$

where $v^2 = \mathbf{E}a_{ij}^2$ [E. Wigner, 1955].

b) Spectral norm $\lambda_{\max}^{(n)} = \max_k\{|\lambda_k^{(n)}|\}$ converges to $2v$ [S. Geman, 1980; Z. Füredi and J. Komlós, 1981, V. Girko, 1988; Z.-D. Bai and Y. Q. Yin, 1988];

$$\lambda_{\max}^{(n)} \to 2v \text{ as } n \to \infty;$$

in particular,

$$\mathbf{P}\left\{\lambda_{\max}^{(n)} \ge 2v(1+x)\right\} \to 0, \ x > 0.$$

I.3 Dilution of random matrices

• Random graphs: symmetric random matrix

$$B_{ij} = \begin{cases} 1, & \text{with probability } \rho/n \\ 0, & \text{with probability } 1 - \rho/n \end{cases}$$

is the adjacency matrix of random graph $G_n(P_n)$ with n vertices and with the edge probability

$$P_n = \rho/n$$

(P. Edős and A. Rényi, 1959; E. Gilbert, 1959)

• Theoretical physics: dilute and sparse disordered systems

- [Rodgers-Bray, 1988]
- [Mirlin-Fyodorov, 1991]

• Neural networks theory

• etcetera, ...

I.4 Semicircle law in dilute RM

In $H^{(n,\rho)}$ a number of bonds (connections) between cites i and j destroyed, the structure of random matrix is changed.

However, if $\rho_n \to \infty$ as $n \to \infty$, the Wigner (or semicircle) law is still valid,

$$\sigma_{n,\rho_n}(\lambda) \to \sigma_W(\lambda)$$

with $\operatorname{supp}(\sigma'_W) = [-2v, 2v]$

- [Rodgers-Bray, 1988]
- [K., Khoruzhenko, Pastur, Shcherbina, 1992]
- [Cazati-Girko, 1992]
- ...

What about $\lambda_{\max}^{(n,\rho)} \to$? and

$$\mathbf{P}\left\{\lambda_{\max}^{(n,\rho)} > 2v(1+x_n)\right\} \ ?$$

II. Critical value for the spectral norm

Theorem [K., *Adv. Probab.* 2001]

If $\rho_n = (\log n)^{1+\beta}$, $\beta > 0$, *then*

$$\mathbf{P}\left\{\lambda_{\max}^{(n,\rho)} > 2v(1+x)\right\} \to 0, \;\; x > 0.$$

If $\rho_n = (\log n)^{1-\beta'}$ *with* $\beta' > 0$, *then*

$$\limsup_{n\to\infty} \lambda_{\max}^{(n,\rho)} = +\infty.$$

Conclusion: the value $\rho_n^* = \log n$ is critical for the asymptotic behavior of $\lambda_{\max}^{(n,\rho_n)}$.

Relation with the properties of large random graphs: the edge probability

$$P_n^* = \frac{\log n}{n}$$

is the critical one (a sharp threshold) with respect to the connectedness of the random graph $G_n(P_n)$.

III.1 Moments of random matrices

Since the works of E. Wigner, the moments

$$M_{2k}^{(n)} = \mathbf{E}\,\frac{1}{n}\,\mathrm{Tr}\left(H^{(n)}\right)^{2k},\ \ k = 0, 1, 2, \ldots$$

have been used to study the moments of $\sigma_n(\lambda)$,

$$M_{2k}^{(n)} = \mathbf{E}\,\frac{1}{n}\sum_{j=1}^{n}\left(\lambda_j^{(n)}\right)^{2k} = \mathbf{E}\ \int \lambda^{2k}\ d\sigma_n(\lambda).$$

In particular, E. Wigner has shown that

$$M_{2k}^{(n)} \to v^{2k}\,\frac{(2k)!}{k!(k+1)!} = v^{2k}\ \mathrm{t}_k,$$

where t_k are the Catalan numbers.

The key idea of S. Geman [*Ann.Probab.*, 1980] inspired by U. Grenander is that the limiting behavior of $\lambda_{\max}^{(n)}$ can be studied by means of the high moments

$$nM_{2k_n}^{(n)},\ \ n, k_n \to \infty.$$

III.2 High moments of Wigner RM

1) $k_n = O(\log n)$ [Geman, 1980; Bai-Yin, 1988]

$$M_{2k_n}^{(n)} \le \left(v^2(1+\varepsilon)\right)^{k_n} \ \mathrm{t}_{k_n}, \quad k_n = O(\log n)$$

implies that

$$\mathbf{P}\left\{\lambda_{\max}^{(n)} > 2v(1+x)\right\} \to 0 \ \text{ as } \ n \to \infty;$$

2) $k_n = O(n^{1/6})$ [Füredi-Komlós, 1981]
$k_n = O(n^{1/2})$, $k_n = o(n^{2/3})$
[Ya. G. Sinai and A. Soshnikov, 1998]

3) $k_n = \chi n^{2/3}$, $\chi > 0$ [A. Soshnikov, 1999]:

$$nM_{2k_n}^{(n)} \to \mathcal{L}(\chi) = \mathcal{L}_{\mathrm{GOE}}(\chi),$$

where $\mathcal{L}(\chi)$ does not depend on the details of the probability distribution of a_{ij}; as a corollary, one gets

$$\mathbf{P}\left\{\lambda_{\max}^{(n)} > 2v\left(1+\frac{y}{n^{2/3}}\right)\right\} \le \mathcal{G}_\chi(y), \ y > 0.$$

The border spectral scale is $n^{-2/3}$.

IV.1 Dilute Wigner RM

Theorem [K., *arXiv*-2011, in preparation] *Let the probability law of* a_{ij} *has a finite support. Then*

$$\mathbf{P}\left\{\lambda_{\max}^{(n,\rho_n)} > 2v\left(1+\frac{y}{n^{2/3}}\right)\right\} \le \mathcal{G}_\chi(y), \ \ y>0$$

for $\rho_n = n^{2/3(1+\gamma)}$ *with any given* $\gamma > 0$.

Main technical results:

A) *If* $\rho_n = n^{2/3(1+\gamma)}$, $\gamma > 0$, *then*

$$\limsup_{n\to\infty} nM_{2k_n}^{(n,\rho_n)} \le \mathcal{L}(\chi), \quad k_n = \chi n^{2/3}.$$

The upper bound $\mathcal{L}$ is universal in the sense that it does not depend on higher moments $V_4, V_6, \ldots$, where $V_{2l} = \mathbf{E}|a_{ij}|^{2l}$, $l \ge 2$.

B) *If* $\rho_n = n^{2/3}$ *and* $k_n = \chi n^{2/3}$, *then*

$$nM_{2k_n}^{(n,\rho_n)} \ge \ell(\chi)\left(1+\chi V_4\right), \ n\to\infty.$$

IV.2 Critical value for border scale

Our results show that the value $\rho_n = n^{2/3}$ represents a critical value for the spectral properties at the border of the spectrum $2v$:

- if the dilution is *weak*, $\rho_n \gg n^{2/3}$, then one can expect that the local spectral properties of Dilute RM are the same as for the Wigner RM ensembles; these properties should be independent on the details of the probability distribution of a_{ij}.

To prove: correlation function of the moments, Moment version of IPR (K. *arXiv, 2010)*

- if the dilution is *moderate*, $\rho_n = O(n^{2/3})$, then the asymptotic behavior of $\lambda_{\max}^{(n)}$ will depend on $V_4 = \mathbf{E}|a_{ij}|^4$. The same can be true for other local spectral characteristics.

- in the case of *strong dilution*, $\rho_n \ll n^{2/3}$, the spectral scale at the border $2v$ changes from $\frac{1}{n^{2/3}}$ to $\frac{\phi(n)}{\rho}$, with $\phi(n) = \log n$ (?)

V. Relations with the Wigner RM

The value of γ in $\rho_n = n^{2/3(1+\gamma)}$ depends on the moments $V_{2l} = \mathbf{E}|a_{ij}|^{2l}$:

$$\text{if } \; V_{12+2\phi} < \infty, \quad \text{then } \; \gamma > \varepsilon = \frac{3}{6+\phi}.$$

Inversely, if $\rho_n = n^{2/3(1+\gamma)}$, then the universal upper bound of $nM_{2k_n}^{(n,\rho_n)}$ exists provided

$$\phi > \frac{3}{\gamma} - 6.$$

For the Wigner ensemble, we have $\rho_n = n$, $\gamma = 1/2$ and then $\phi > 0$, in accordance with the following generalization of earlier results [A. Soshnikov, 1999];

Theorem [K. 2012] *If* $V_{12+2\delta}$ *exists for any* $\delta > 0$*, then for the Wigner RM,*

$$\lim_{n\to\infty} n\, M_{2k_n}^{(n)} = \mathcal{L}_{GOE}(\chi), \quad k_n = \chi n^{2/3},$$

where $\mathcal{L}_{GOE}$ *(or* $\mathcal{L}_{GUE}$*) does not depend on the moments of* V_{2l}*,* $l = 2, ..., 6$ *and on* $V_{12+2\delta}$*.*

VI.1 Proof of the upper bound

The proof is based on the method of paper [K., *Rand. Oper. Stoch. Eqs.* 2012], where a modified and improved version of the approach by Ya.G.Sinai and A. Soshnikov completed in [K. and Vengerovsky, *arXiv*, 2008] is presented.

Start point: E. Wigner's representation of traces

$$nM_{2k} = \sum_{i_0,\ldots,i_{2k-1}} \mathbf{E} \left\{ H_{i_0,i_1} \cdots H_{i_{2k-1},i_0} \right\}$$

as a sum over $2k$-step trajectories

$$\mathcal{I}_{2k} = (i_0, i_1, i_2, \ldots, i_{2k-2}, i_{2k-1}, i_0).$$

The family $\{\mathcal{I}_{2k}\}$ can be separated into the classes of equivalence determined by the number $\mathcal{K}$ of self-intersections of the trajectories.

When $\mathcal{K} = 0$, the classes are described by the family $\mathcal{D}_{2k}$ of the Dyck paths: discrete simple walks of $2k$ steps in the upper half-plane that start and end at 0. These are equivalent to the rooted half-plane trees. The cardinality $|\mathcal{D}_{2k}|$ is given by the Catalan number t_k.

VI.2 Technical questions

- Wigner RM, Sinai-Soshnikov approach: the study of simple self-intersections (open ones; V_4-direct); vertex of maximal exit degree β;

- K., Vengerovsky: proper and imported cells at β; Brocken-Tree-Structure instants;

- K. *Rand.Oper.Stoch.Eqs.*: V_4-direct and inverse edges; generalization to the case of V_{2k}

- Dilute RM, K. 2012: detailed study of the vertex β of maximal exit degree D;

$$D = d_1 + \ldots + d_L, \quad \bar{d}_L = (d_1, \ldots, d_L). \quad (A)$$

The following statement improves the tools used by Ya. G. Sinai and A. Soshnikov.

D-lemma. *Denote by $\mathcal{T}_k^{(u)}(\bar{d}_L)$ the family of Catalan trees of height u that have L vertices of exit degrees $\bar{d}_L$ (A). Then*

$$\sum_{u=1}^{k} e^{\chi u/\sqrt{k}} \; |\mathcal{T}_k^{(u)}(\bar{d}_L)| \;\leq\; L e^{-\eta D}\, B(\chi)\, \mathrm{t}_k,$$

where $\eta = \ln(4/3)$ and $B(\chi)$ is related with the Brownian bridge.

VII.1 Tree-type walks with multiple edges

Each plane tree generates, by the chronological run over it, a walk of $2k$ steps such that each edge is passed exactly <u>two times</u> (there and back). The number of these *Catalan walks* is

$$\mathrm{t}_k = \frac{(2k)!}{k!\,(k+1)!}\,, \quad k \geq 0.$$

Lemma [K.,*arXiv*, 2012] Consider the family of Catalan-type walks of $2k$ steps such that there exists exactly one special edge passed <u>four times</u>. Then its cardinality is given by

$$\mathrm{t}_k^{(2)} = \frac{(2k)!}{(k-2)!\,(k+2)!}, \; k \geq 2,$$

with obvious equalities $\mathrm{t}_0^{(2)} = \mathrm{t}_1^{(2)} = 0.$

Remark. The cardinality of Catalan walks with one colored edge is obviously equal to

$$\mathrm{t}_k^{(1)} = \frac{(2k)!}{(k-1)!\,(k+1)!}, \; k \geq 1.$$

VII.2 Bound from below

Relation

$$\mathrm{t}_k^{(2)} = \frac{(2k)!}{(k-2)!\,(k+2)!} = \left(k - \frac{3k}{k+2}\right)\ \mathrm{t}_k$$

shows that $\mathrm{t}_k^{(2)} \ge k\,\mathrm{t}_k/2$, $k \ge 4$. This implies the lower bound for the moments of $H^{(n,\rho_n)}$.

Indeed,

$$\mathbf{E}\left(H^{(n,\rho_n)}\right)^{2k} \ge n\mathrm{t}_k\, V_2^2 + nV_2^{k-2} \cdot \frac{V_4}{\rho} \cdot \mathrm{t}_k^{(2)}$$

$$\ge n\mathrm{t}_k\, V_2^2 \left(1 + \frac{k\, V_4}{2\rho\, V_2^2}\right)\ .$$

Therefore, if $k = \chi n^{2/3}$ and $\rho = n^{2/3}$, then the estimate from below explicitly contains a non-vanishing term $\chi V_4/2V_2^2$.

This means that the estimate from above of the moments of the dilute random matrices in the asymptotic regime $\rho = n^{2/3}$ is crucially different from that in the regime $\rho = o(n^{2/3})$.

VI.3 Recurrent relations for $\mathbf{t}_k^{(2)}$

The Catalan numbers t_k are determined by recurrence

$$\mathrm{t}_k = \sum_{u+v=k-1} \mathrm{t}_u\, \mathrm{t}_v, \quad k \geq 1,$$

$t_0 = 1$; it can be obtained with the help of the <u>reduction of the ground step</u> procedure.

A simple reasoning shows that

$$\mathrm{t}_k^{(2)} = \sum_{u+v+r+s=k-2} (2u+1)\ \mathrm{t}_u\, \mathrm{t}_v\, \mathrm{t}_r\, \mathrm{t}_s\,,$$

for $k \geq 2$. The use of the generating function of t_k leads to the explicit expression for $\mathrm{t}_k^{(2)}$.

Several first values of $\mathrm{t}_k^{(2)} = \dfrac{(2k)!}{(k-2)!\,(k+2)!}$ are as follows,

$$1,\ 6\,,28,\ 120,\ 495, \ldots$$

At present time, the N. Sloan's encyclopedia of integer sequences (OEIS) says nothing about this sequence.

VI.4 More about the sequences $\mathbf{t}_k^{(m)}$

Let us denote by $\mathrm{t}_k^{(m)}$, $m \geq 1$ the set of even closed tree-type walks of $2k$ steps such that all edges are passed two times (there and back) and there exists one special edge passed $2m$ times.

Question: what is the explicit form of $\mathrm{t}_k^{(3)}$?

$$\mathrm{t}_k^{(m)} = \sum_{u+v_1+\ldots+v_{2m-1}=k-m} (2u+1)\mathrm{t}_u\mathrm{t}_{v_1}\cdots\mathrm{t}_{v_{2m-1}}.$$

Answer: it is not hard to show that

$$\mathrm{t}_k^{(3)} = \frac{(2k)!}{(k-3)!\,(k+3)!}, \quad k \geq m \geq 3.$$

It is natural to assume that for any $m \geq 1$,

$$\mathrm{t}_k^{(m)} = \frac{(2k)!}{(k-m)!\,(k+m)!}, \quad k \geq m.$$

VII.5 Why to study $\mathbf{t}_k^{(m)}$?

In the regime $\rho = n^{2/3}$, the estimate from below of the moments involves the terms

$$n\, M_{2k}^{(n,\rho)} \ge n \mathrm{t}_k\, V_2^2 \left(1 + \frac{k\, V_4}{2\rho\, V_2^2} + \frac{k\, V_6}{6\rho^2\, V_2^3} + \ldots\right) \qquad (B)$$

for sufficiently large values of k because

$$\mathrm{t}_k^{(3)} = \frac{(2k)!}{(k-3)!\,(k+3)!} = \mathrm{t}_k \left(k - 8 - \frac{36k+48}{k^2+5k+6}\right).$$

Expression of the form $\frac{k\, V_6}{\rho^2\ V_2^3}$ means that the terms with V_6 should disappear from the limiting expression for $n\, M_{2k}^{(n,\rho)}$. The same could be true for the terms with $V_8, V_{10}, \ldots$.

Conjecture. The limiting expression for $nM_{2k}^{(n,\rho)}$ with $\rho_n = n^{2/3}$ contains the Wigner-GOE part (Wigner-GUE part for the case of Hermitian matrices) and the terms that involve V_4, <u>but not</u> $V_{2k},\ k \ge 3$.

VIII.1 Beyond the threshold $n^{2/3}$

Let us try to imagine the picture for the strong dilution regime $\rho_n \ll n^{2/3}$. One can expect the following phenomena in the walks:

- the walks that have self-intersections of degree $\kappa = 3$ disappear from the limiting $n\, M_{2k}^{(n,\rho)}$;

- the walks that have simple self-intersections with broken tree structure disappear from the limiting $n\, M_{2k}^{(n,\rho)}$;

  Consequence: the difference between real symmetric and hermitian cases vanishes;

- if our V_4-conjecture is true, then the walks that have multiple edges V_{2l} with $l \geq 3$ disappear from the limiting expression for $n\, M_{2k}^{(n,\rho)}$.

One could assume that the leading contribution to $n\, M_{2k}^{(n,\rho)}$ is given by the tree-type walks with simple self-intersections only ($\kappa = 2$) that have 2- and 4-multiple edges.

VIII.2 Basic walks for moments

Instead of the Catalan walks of $2k$ steps, where each edge is passed two times (there and back), the walks with 2- and 4-multiple edges could play the role of the basic walks. So, the number of such basic walks is given by the number $\mathrm{T}_k = \mathrm{T}_k^{(2,4)}$ of these $(2,4)$-*Catalan* walks.

We can write that $T_k = R_k^{(0)}(\rho)$, where

$$R_k^{(0)} = a \sum_{u=0}^{k-1} R_{k-1-u}^{(0)} R_u^{(0)} + \frac{b}{k} \sum_{u=0}^{k-2} R_{k-2-u}^{(1)} R_u^{(1)}$$

with $a = V_2 = v^2$ and $b = \chi V_4$.

This recurrent relation resembles the one for the semicircle moments $v^{2k}\,\mathrm{t}_k$, but is in fact (much) more complicated.

Finally, to find the limiting expression for $\mathcal{L}_{\mathrm{DRM}}$, one could try with

$$\lim_{n,k\to\infty} \quad nR_k^{(0)}(\rho), \quad \rho = \chi k.$$

VIII.3 Equations for $R_k^{(m)}$

For $k \geq 1$ and $m \geq 1$, we have

$$R_k^{(m)} = R_k^{(m-1)} + a \sum_{u=0}^{k-1} R^{(0)}_{k-1-u} R_u^{(m)}$$

$$+ \frac{b}{k} \sum_{u=0}^{k-2} R^{(1)}_{k-2-u} R_u^{(m+1)},$$

where $a = v^2 = V_2$ and $b = \chi V_4$.

In other terms,

$$R_k^{(m)} = \sum_{r=0}^{k} (r+1)(r+2)\cdots(r+m)\, S(k,r);$$

the numbers $S(k,r)$, $1 \leq r \leq k$ are uniquely determined by recurrence

$$S(k,r) = a \sum_{u=0}^{k-r} \sum_{v=0}^{u} S(u,v)\, S(k-u-1, r-1)$$

$$+ \frac{b}{k} \sum_{u=0}^{k-r} (r-1) \sum_{v=0}^{u} (v+1)\, S(u,v)\, S(k-u-2, r-2).$$