Packing dimension profiles & Lévy processes

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(joint work with René Schilling & Yimin Xiao)

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Porquerolles Islands, France
June 13–17, 2011

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- $\exists$ a different, more complicated, formula when $d = 1$ [K–Xiao, 2011].
- Open Problem: What is $\|\dim_{\rm p}(W(F) \cap E)\|_{L^\infty(\mathrm{P})}$?

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- Question. What is $\dim_{\mathrm{p}} X(F)$?
- $\dim_{\mathrm{p}} X(F)$ is known in a few cases.

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- An example of the more interesting case 2: Let $\{X(t)\}_{t \geqslant 0}$ denote an isotropic stable-α Lévy process on $\mathbf{R}^d$ with $d \geqslant \alpha$. Then,

$$\dim_{\mathrm{p}} X(F) = \alpha \dim_{\mathrm{p}} F \qquad \text{(Perkins and Taylor, 1987).}$$

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- Question. What happens when $d < \alpha$?
- Answer known only when $d = 1$, $\alpha = 2$ (Xiao, 1997).

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On 1-D Brownian motion

If $X :=$ d-D Brownian motion and $d \geqslant 2$, then $\dim_{\rm p} X(F) = 2\dim_{\rm p} F$ a.s.

Theorem (Xiao, 1997)

If $X :=$ d-D Brownian motion, then

$$\dim_{\rm p} X(F) = 2\mathrm{Dim}^{\rm FH}_{d/2} F,$$

where $\mathrm{Dim}^{\rm FH}_{\beta} :=$ β-dimensional Falconer–Howroyd packing dimension profile (Falconer and Howroyd, 1997).

A consequence of the ensuing theory

Corollary (K, Schilling, and Xiao, 2011)

If $X :=$ d-D isotropic stable-α Lévy process, then

$$\dim_{\mathrm{P}} X(F) = \alpha \mathrm{Dim}^{\mathrm{FH}}_{d/\alpha} F.$$

The method of proof is different from that of Xiao (1997), in part because stable processes do not have continuous trajectories.

Another consequence of the ensuing theory

Corollary (K, Schilling, and Xiao, 2011)

If S := *subordinator with Laplace functional* Φ, *i.e.,* $\mathrm{E}\exp(-\lambda S(t)) = \exp(-t\Phi(\lambda))$, *and* F=*compact, then*

$$\overline{\dim}_{\mathsf{M}} S(F) = \sup\left\{\eta > 0 : \overline{\lim_{\lambda\uparrow\infty}}\ \lambda^{\eta} \cdot \inf_{\nu\in M_1(F)} \iint \mathrm{e}^{-|t-s|\Phi(\lambda)}\, \nu(\mathrm{d}s)\, \nu(\mathrm{d}t) = 0\right\}.$$

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- Can regularize to obtain $\dim_{\mathrm{p}} S(F)$.
- Simplifies when F is nice; e.g.,

$$\overline{\dim}_{\mathrm{M}} S[0\,,1] = \sup\left\{\eta > 0 : \varliminf_{\lambda\uparrow\infty} \frac{\Phi(\lambda)}{\lambda^\eta} = \infty\right\}$$

(Fristedt and Taylor, 1992; Bertoin, 1999).

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A final consequence of the ensuing theory

Corollary (K, Schilling, and Xiao, 2011)

Let $S :=$ *subordinator with Laplace exponent* Φ. *Then* $^{\forall} s \geq 1/2$,

$$\mathrm{Dim}_s^{\mathrm{FH}} S(\mathbf{R}_+) = s \times \left(1 - \lim_{\lambda \uparrow \infty} \frac{1}{\log \lambda} \log \int_1^{\lambda} \frac{dx}{\Phi(x^{1/s})} \right).$$

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- The zero-set of every Markov process is equal to the closure of $S(\mathbf{R}_+)$ for *an explicit* subordinator S [Maisonneuve, 1974].

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$$\overline{\mathrm{Dim}}_\kappa F := \varlimsup_{\epsilon\downarrow 0} \frac{1}{\log \epsilon} \log \inf_{\nu \in M_1(F)} \iint \kappa_\epsilon(|t-s|)\nu(ds)\nu(dt).$$

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- Define the *packing dimension profile of* F:

$$\mathrm{Dim}_\kappa F := \inf \sup_{n\geqslant 1} \overline{\mathrm{Dim}}_\kappa F_n,$$

where the inf is over all covers of F by bounded Borel sets $F_1, F_2, \ldots$.

Main result

Theorem (K, Schilling, and Xiao, 2011)

Let $\{X(t)\}_{t\geqslant 0} :=$ *a Lévy process in* $\mathbf{R}^d$. *Then*

$$\overline{\dim}_{\mathrm{M}} X(F) = \overline{\mathrm{Dim}}_\kappa F, \qquad \dim_{\mathrm{P}} X(F) = \mathrm{Dim}_\kappa F.$$

On the proof (the upper bound)

- Let P_x denote the law of the Lévy process started at x, i.e., $x + X(\bullet)$, and define

$$\mathscr{P} := \int_{\mathbf{R}^d} \mathrm{P}_x \, \mathrm{d}x.$$

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Proposition

$$\mathscr{P}\left\{\inf_{t \in F} \|X(t)\| \leqslant \epsilon\right\} \leqslant \frac{2^{d+1}(2\epsilon)^d}{\inf_{\nu \in M_1(F)} \iint \kappa_\epsilon(|t-s|)\nu(ds)\nu(dt)}.$$

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- $\therefore$ we have an upper bound for $\mathrm{E}(\mathsf{Leb}(X(F)^{\epsilon}))$.
- Let $\mathscr{K}_G(\epsilon)$:= max. no. $m \geqslant 1$ of points $x_1, \ldots, x_m \in G$ with $\min_{j\neq k}\|x_j - x_k\| \geqslant \epsilon$ (Kolmogorov capacitance).

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- $c\epsilon^d \mathscr{K}_G(\epsilon) \leqslant \mathsf{Leb}(G^{\epsilon})$ because $^{\exists}k := \mathscr{K}_G(\epsilon)$ points $x_1, \ldots, x_k \in G$ such that $B(x_j, \epsilon/2)$ are disjoint and $G^{\epsilon} \supset \cup_{j=1}^{k} B(x_j, \epsilon/2)$.

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- $\Rightarrow$ an upper bound for $\mathrm{E}[\mathscr{K}_{X(F)}(\epsilon)]$.

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- $\Rightarrow$ an upper bound for $\mathrm{E}[\mathscr{K}_{X(F)}(\epsilon)]$. If $\mathrm{E}[\mathscr{K}_{X(F)}(\epsilon)] = O(\epsilon^{-\eta})$ then [Borel–Cantelli] $\mathscr{K}_{X(F)}(\epsilon) = O(\epsilon^{-\eta+o(1)})$ a.s. $\Rightarrow$ upper bound for $\overline{\dim}_{\mathrm{M}} X(F)$ $\Rightarrow$ U.B. for $\dim_{\mathrm{P}} X(F)$ by "regularization."

On the proof (the lower bound)

- Use density: For all $\nu \in M_1(F)$ define $m := \nu \circ X^{-1}$ $[m \in M_1(X(F))]$

$$\mathrm{E} \iint \frac{\mathbf{1}_{\{\|x-y\| \leqslant \epsilon\}}}{\epsilon^\eta} m(\mathrm{d}x) m(\mathrm{d}y) = \iint \frac{\kappa_\epsilon(|t-s|)}{\epsilon^\eta} \nu(\mathrm{d}s) \nu(\mathrm{d}t).$$

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- Optimize over $\nu \in M_1(F)$, let $\epsilon \downarrow 0$ using Fatou's lemma.

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- Therefore,

$$\iint \kappa_\epsilon(|t-s|)\nu(\mathrm{d}s)\nu(\mathrm{d}t) \asymp \epsilon^d \iint \left(|t-s|^{-d/\alpha} \wedge \epsilon^{-d}\right) \nu(\mathrm{d}s)\nu(\mathrm{d}t).$$

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- Apply a proposition of K–Xiao (2010) to relate this to Falconer–Howroyd potentials.