Minkowski content and fractal curvature measures for self-conformal sets

1. Motivation and Aim

- The Minkowski content can be viewed as a measure of lacunarity for a fractal set, which enables one to distinguish between sets of equal box dimension (cf. [Man95]).
- Results on the existence of the Minkowski content are used in studying the distribution of eigenvalues of the Laplacian on domains with fractal boundaries (cf. [LP93]).
- Fractal curvature measures are refinements of the notion of Minkowski content and provide a notion of curvature for fractal sets.

Determining classes of sets for which the Minkowski content and the fractal curvature measures exist and extending existing results for self-similar sets to self-conformal sets.

2. Definitions and Notation

- λ^1 and λ^0: one- and zero-dimensional Lebesgue measure.
- $Y_\varepsilon := \{y \in \mathbb{R} \mid \text{dist}(y, F) \leq \varepsilon\}$ for $Y \subset \mathbb{R}$ and $\varepsilon > 0$.
- ∂Y: boundary of $Y \subset \mathbb{R}$.

Definition 1 (Minkowski content) Let $Y \subset \mathbb{R}$ be compact with box dimension δ. The *upper* and *lower Minkowski contents* of Y are defined by

$$\overline{\mathcal{M}}(Y) := \limsup_{\varepsilon\to 0} \varepsilon^{\delta-1}\lambda^1(Y_\varepsilon) \quad \text{and} \quad \underline{\mathcal{M}}(Y) := \liminf_{\varepsilon\to 0} \varepsilon^{\delta-1}\lambda^1(Y_\varepsilon).$$

If $\overline{\mathcal{M}}(Y) = \underline{\mathcal{M}}(Y)$, then the common value is called the *Minkowski content* of Y and is denoted by $\mathcal{M}(Y)$. If it exists, the *average Minkowski content* is the limit

$$\widetilde{\mathcal{M}}(Y) := \lim_{T\searrow 0} |\ln T|^{-1} \int_T^1 \varepsilon^{\delta-2}\lambda^1(Y_\varepsilon)\mathrm{d}\varepsilon.$$

Definition 2 (Fractal curvature measures) Let $Y \subset \mathbb{R}$ be compact. If the weak limits

$$C_1^f(Y,\cdot) := \operatorname*{w\text{-}lim}_{\varepsilon\to 0} \varepsilon^{\delta-1}\lambda^1(Y_\varepsilon \cap \cdot) \quad \text{and} \quad C_0^f(Y,\cdot) := \operatorname*{w\text{-}lim}_{\varepsilon\to 0} \varepsilon^{\delta}\lambda^0(\partial Y_\varepsilon \cap \cdot)/2$$

exist, $C_1^f(Y,\cdot)$ and $C_0^f(Y,\cdot)$ are called the *1-st* and *0-th fractal curvature measures* of Y. If

$$\widetilde{C}_1^f(Y,\cdot) := \operatorname*{w\text{-}lim}_{T\searrow 0} |\ln T|^{-1} \int_T^1 \varepsilon^{\delta-2}\lambda^1(Y_\varepsilon \cap \cdot)\mathrm{d}\varepsilon \quad \text{and}$$

$$\widetilde{C}_0^f(Y,\cdot) := \operatorname*{w\text{-}lim}_{T\searrow 0} |\ln T|^{-1} \int_T^1 \varepsilon^{\delta-1}\lambda^0(\partial Y_\varepsilon)\mathrm{d}\varepsilon/2$$

exist, then they are called the *1-st* and *0-th average fractal curvature measures* of Y.

Definition 3 (Self-conformal set) Let $\emptyset \neq X \subset \mathbb{R}$ denote a compact interval and let $\phi_1, \dots, \phi_N : X \to X$ be differentiable with α-Hölder continuous derivatives, s. t. $0 < |\phi_1'|, \dots, |\phi_N'| < 1$. Assume that $\Phi := \{\phi_1, \dots, \phi_N\}$ satisfies the OSC with open set $O :=$ interior(X), i. e. $\bigcup_{i=1}^N \phi_i(O) \subseteq O$ and $\phi_i(O) \cap \phi_j(O) = \emptyset$ for $i \neq j$. The unique non-empty compact invariant set F associated to Φ is called the *self-conformal set* associated to Φ, provided $\lambda^1(F) = 0$.

- (Σ^∞, σ): full shift-space on N symbols.
- $\xi\colon \Sigma^\infty \to \mathbb{R}$, $\omega = \omega_1\omega_2 \cdots \mapsto -\ln|\phi_{\omega_1}'(\sigma\omega)|$ geometric potential function to Φ.

Definition 4 (lattice, non-lattice) Φ is called *lattice*, if there exists a $\psi \in \mathcal{C}(\Sigma^\infty)$ s. t. the range of $\xi - \psi + \psi\circ\sigma$ is contained in a discrete subgroup of $\mathbb{R}$. If no such ψ exists, then Φ is called *non-lattice*.

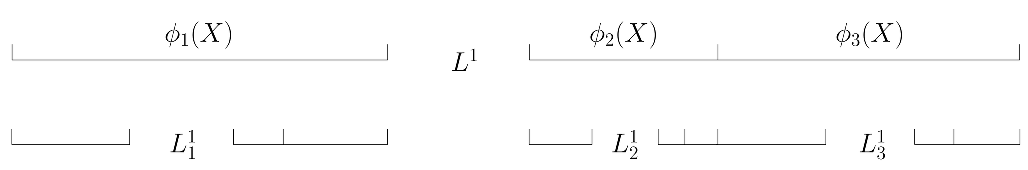

FIGURE 1: Labelling of the gaps.

- δ: box dimension of F.
- $h_{\mu_{-\delta\xi}}$: measure theoretical entropy of σ w. r. t. the unique σ-invariant Gibbs-measure for the potential function $-\delta\xi$.
- ν: the δ-conformal measure associated to Φ, i. e. the unique probability measure supported on F which satisfies

$$\nu(\phi_i X \cap \phi_j X) = 0 \text{ for } i \neq j \quad \text{and} \quad \nu(\phi_i B) = \int_B |\phi_i'|^\delta \mathrm{d}\nu$$

for all $i \in \{1, \dots, N\}$ and all Borel-sets $B \subseteq X$.
- $L^1, \dots, L^Q$: the connected components of $X \setminus \Phi X$.
- $L_\omega^i := \phi_\omega(L^i)$, where $\phi_\omega := \phi_{\omega_1} \circ \cdots \circ \phi_{\omega_n}$ for $\omega = \omega_1 \cdots \omega_n \in \{1, \dots, N\}^n$ and $i \in \{1, \dots, Q\}$.

3. Results

Theorem 1 (Self-conformal sets – fractal curvature measures)
(i) The average fractal curvature measures always exist. Additionally,

$$\widetilde{C}_0^f(F,\cdot) = \frac{2^{-\delta}c}{h_{\mu_{-\delta\xi}}}\cdot\nu(\cdot) \quad \text{and} \quad \widetilde{C}_1^f(F,\cdot) = \frac{2^{1-\delta}c}{(1-\delta)h_{\mu_{-\delta\xi}}}\cdot\nu(\cdot), \tag{1}$$

where the constant $c > 0$ is given by the well-defined limit

$$c := \lim_{n\to\infty} \sum_{i=1}^{Q} \sum_{\omega\in\Sigma^n} |L_\omega^i|^\delta.$$

(ii) If Φ is non-lattice, then both the 0-th and 1-st fractal curvature measures exist and $C_k^f(F,\cdot) = \widetilde{C}_k^f(F,\cdot)$ for $k \in \{0,1\}$.
(iii) If Φ is lattice, then neither the 0-th nor the 1-st fractal curvature measure exists.

Remark 1 (Self-conformal sets – Minkowski content) Clearly, $\widetilde{\mathcal{M}}(F) = \widetilde{C}_1^f(F,\mathbb{R})$ and in the non-lattice case $\mathcal{M}(F) = C_1^f(F,\mathbb{R})$. Further, there are lattice self-conformal sets for which the Minkowski content does exist (cf. Example 1).

Remark 2 (Self-similar sets) Assume that $\phi_1, \dots, \phi_N$ are similarities with contraction ratios $r_1, \dots, r_N$, then Eq. (1) simplifies to

$$\widetilde{C}_0^f(F,\cdot) = \frac{2^{-\delta}\sum_{i=1}^{Q-1}|L^i|^\delta}{-\delta\sum_{i\in\Sigma}\ln(r_i)r_i^\delta}\cdot\nu(\cdot) \quad \text{and} \quad \widetilde{C}_1^f(F,\cdot) = \frac{2^{1-\delta}\sum_{i=1}^{Q-1}|L^i|^\delta}{(\delta-1)\delta\sum_{i\in\Sigma}\ln(r_i)r_i^\delta}\cdot\nu(\cdot). \tag{2}$$

Alternative formulae to Eq. (2) were obtained in [Win08]. For the Minkowski content the resulting formulae coincide with those presented in [Fal95, LvF06]. In the lattice case the Minkowski content never exists.

Theorem 2 ($\mathcal{C}^{1+\alpha}$ images – fractal curvature measures) *Let $F \subset \mathbb{R}$ be self-similar. Let $\mathcal{U} \supset X$ be a connected open neighbourhood of X and $g\colon \mathcal{U} \to \mathbb{R}$ be a $\mathcal{C}^{1+\alpha}(\mathcal{U})$ map with $\alpha > 0$, for which $|g'|$ is bounded away from 0. The average fractal curvature measures of both F and $g(F)$ exist. Moreover, they are absolutely continuous and for $k \in \{0,1\}$ their Radon-Nikodym derivatives are given by*

$$\frac{\mathrm{d}\widetilde{C}_k^f(g(F),\cdot)}{\mathrm{d}\widetilde{C}_k^f(F,\cdot)\circ g^{-1}} = |g'\circ g^{-1}|^\delta.$$

In the non-lattice case the same holds for $C_k^f(F,\cdot)$ and $C_k^f(g(F),\cdot)$.

For an extension of Theorem 2 to higher dimensions cf. [FK].

Example 1 Let $F \subseteq [0,1]$ be the middle third Cantor set. Let $\widetilde{g}\colon \mathbb{R} \to \mathbb{R}$ denote the devil's staircase function defined by $\widetilde{g}(r) := \nu((-\infty, r])$, define $g\colon [-1,\infty) \to \mathbb{R}$ by

$$g(x) := \int_{-1}^x (\widetilde{g}(y)+1)^{-\ln 3/\ln 2}\mathrm{d}y.$$

Then $\underline{\mathcal{M}}(g(F)) = \overline{\mathcal{M}}(g(F))$, although $\underline{\mathcal{M}}(F) < \overline{\mathcal{M}}(F)$.

(joint work with M. Kesseböhmer)

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