Multi-operator scaling random fields

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supported by the ANR Project MATAIM.

Definition 1 *A random field* $X = (X(x))_{x\in\mathbb{R}^d}$ *is* **self-similar** *of order* H *if*

$$\forall\varepsilon > 0,\ (X(\varepsilon x))_{x\in\mathbb{R}^d} \stackrel{(d)}{=} \varepsilon^H (X(x))_{x\in\mathbb{R}^d}.$$

- Global Property.

- Let X be a non degenerate continuous Gaussian random field self-similar of order H with stationary increments.

$$\mathbb{E}\big((X(x+y)-X(x))^2\big) = \|y\|^{2H}\mathbb{E}\big(X(y/\|y\|)^2\big),\quad y\neq 0.$$

- **Directional pointwise Hölder exponent**

Let $\theta\in\mathbb{S}^{d-1}$ s.t. $X(\theta)\neq 0$. Then,

$$H_X(x,\theta) = \sup\left\{\gamma\in\mathbb{R},\ \lim_{t\to 0}\frac{X(x+t\theta)-X(x)}{|t|^\gamma} = 0\right\} = H \quad \text{a.s.}$$

- **Pointwise Hölder exponent**

$$H_X(x) = \sup\left\{\gamma\in\mathbb{R},\ \lim_{y\to 0}\frac{X(x+y)-X(x)}{\|y\|^\gamma} = 0\right\} = H \quad \text{a.s.}$$

Definition 2 (Kolmogorov 1940 / Mandelbrot, Van Ness 1968) *The FBM B_H of exponent $H \in (0,1)$ is the real centered Gaussian random field such that for every $x, y \in \mathbb{R}^d$,*

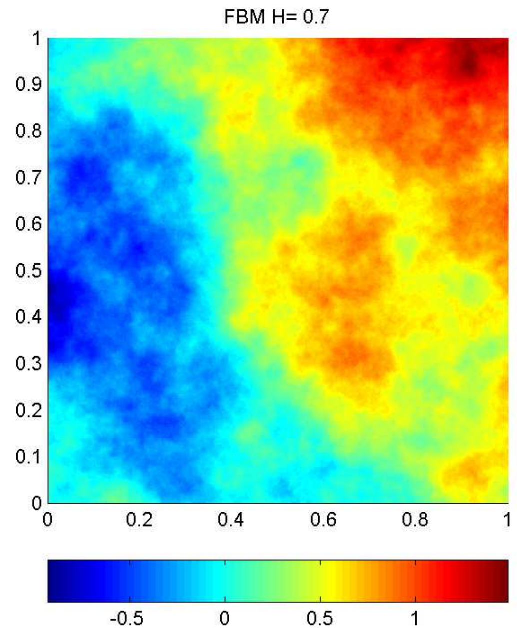
$$\mathbb{E}\left(B_H(x)\, B_H(y)\right) = \frac{1}{2}\left[\|x\|^{2H} + \|y\|^{2H} - \|x-y\|^{2H}\right].$$

☞ only isotropic H-self-similar Gaussian field with stationary increments, up to a deterministic multiplicative constant.

☞ **Harmonizable Representation**

$$B_H(x) = \frac{1}{C_H}\Re\int_{\mathbb{R}^d} \frac{\mathrm{e}^{ix\cdot\xi}-1}{\|\xi\|^{H+d/2}}\, W_2(d\xi)\,,$$

with $W_2(d\xi)$ an isotropic complex Gaussian random measure.



FBM with $d = 2, H = 0.7$

(Stein Method)

Definition 3 (Benassi, Jaffard, Roux/ Peltier, Lévy Véhel) *A random field* $X = (X(x))_{x\in\mathbb{R}^d}$ *is* **locally asymptotically self-similar** *(lass) of order* $h(x)$ *at point* x *if*

$$\lim_{\varepsilon\to 0_+}\left(\frac{X(x+\varepsilon u)-X(x)}{\varepsilon^{h(x)}}\right)_{u\in\mathbb{R}^d} \overset{(d)}{=} (Z_x(u))_{\in\mathbb{R}^d}$$

with Z_x *a non degenerate random field.*

- ☛ H-self-similar $\Longrightarrow$ Lass at 0 of order H with $Z_0 = X$.
- ☛ Z_x is self-similar of order $h(x)$.
- ☛ X continuous lass at any $x \Longrightarrow$ for almost x, Z_x has stationary increments (FALCONER).

Multifractional Brownian Motion

$$B_h(x) = \frac{1}{C_{h(x)}} \Re \int_{\mathbb{R}^d} \frac{e^{ix\cdot\xi} - 1}{\|\xi\|^{h(x)+d/2}} \, W_2(d\xi) \,,$$

with $h : \mathbb{R}^d \to (0,1)$ a Lipschitz function.

☛ Lass at point x of order $h(x)$ with tangent field $Z_x = B_{h(x)}$ a FBM of order $h(x)$.

☛ **Pointwise Hölder exponent:**

$$H_{B_h}(x) = h(x) \quad \text{a.s.}$$

☛ **Directional pointwise Hölder exponent:** for any $\theta \in \mathbb{S}^{d-1}$,

$$H_{B_h}(x,\theta) = h(x) \quad \text{a.s.}$$

Definition 4 (Biermé, Meerschaert, Scheffler 2007) *A random field* $(X(x))_{x\in\mathbb{R}^d}$ *is* **operator scaling** *for a matrix* $E \in \mathcal{M}_d(\mathbb{R})$ *if*

$$\forall c > 0,\ \left(X\left(c^E x\right)\right)_{x\in\mathbb{R}^d} \stackrel{(d)}{=} c\,(X(x))_{x\in\mathbb{R}^d}$$

with $c^E = \exp(E\ln(c))$.

Remark If $E = \mathrm{diag}(\lambda_1,\ldots,\lambda_d)$, then, $X\left(c^E x\right) = X\left(c^{\lambda_1}x_1,\ldots,c^{\lambda_d}x_d\right)$.

☛ H-self-similar $\Longleftrightarrow$ Operator scaling for $E = Id/H$.

☛ **Example 1:** The fractional Brownian sheet

$$\widetilde{B}(x) = \Re\int_{\mathbb{R}^d}\prod_{i=1}^{d}\frac{\mathrm{e}^{ix_j\xi_j}-1}{|\xi_j|^{H_j+1/2}}W_2(d\xi),\quad x\in\mathbb{R}^d,$$

is operator scaling for $E = \mathrm{diag}(1/H_1,\ldots,1/H_d)$.

☛ Consider X a continuous operator scaling field for E. Let $u \in \mathbb{R}^d \backslash \{0\}$ and $\lambda \in \mathbb{R}\backslash\{0\}$ s.t. $Eu = \lambda u$.

- $(X(tu))_{t\in\mathbb{R}}$ is self-similar with index $1/\lambda$, i.e.

$$\forall c > 0, (X(ctu))_{t\in\mathbb{R}} \overset{(d)}{=} c^{1/\lambda} (X(tu))_{t\in\mathbb{R}},$$

The scaling property may depend on the direction u.

- X Gaussian with stationary increments s.t. $X(u) \neq 0 \Longrightarrow H_X(x, u) = 1/\lambda$ a.s.

The directional Hölder regularity may depend on the direction u.

☛ **Example 2:** BIERMÉ, MEERSCHAERT, SCHEFFLER 2007

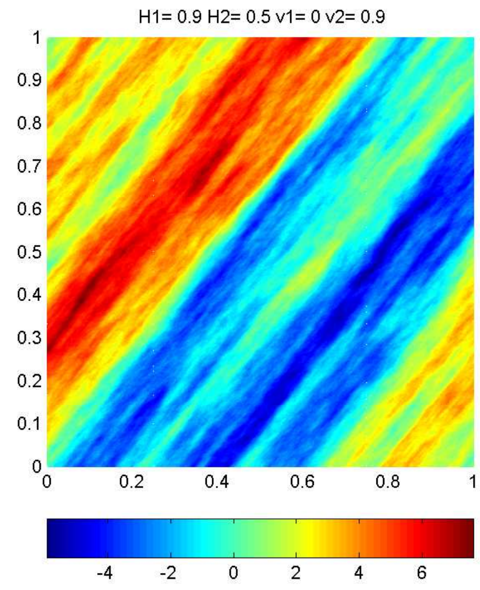
$$X_{\alpha,\psi}(x) = \Re \int_{\mathbb{R}^d} \left(\mathrm{e}^{ix\cdot\xi} - 1\right) \psi(\xi)^{-1-\mathrm{tr}(E)/\alpha} W_\alpha(d\xi)$$

with W_α an α-stable isotropic complex random measure and ψ E^t-homogeneous, i.e. s.t.

$$\forall c > 0, \forall x \in \mathbb{R}^d,\ \psi\left(c^{E^t} x\right) = c\psi(x).$$

- $E = Id/H$ and $\psi = \|\cdot\|^H \Longrightarrow X_{2,\psi} =$ FBM of exponent H.
- X well-defined and stoch. cont. $\Longleftrightarrow \min_{\lambda \in \mathrm{Sp}E} \Re(\lambda) > 1$.
- Stationary increments.
- $H_X(x, u)$ may depend on the direction u but not on x.

$\rightsquigarrow$ Sample paths regularity: BIERMÉ, MEERSCHAERT, SCHEFFLER / BIERMÉ, L. / CLAUSEL, VÉDEL.



Definition 5 (Biermé, L., Scheffler, 2011) *A random field* $X = (X(x))_{x \in \mathbb{R}^d}$ *is* **locally asymptotically operator scaling** *(laos) at point* x *for the matrix* $E(x)$ *if*

$$\lim_{\varepsilon \to 0_+} \left(\frac{X\left(x + \varepsilon^{E(x)} u\right) - X(x)}{\varepsilon} \right)_{u \in \mathbb{R}^d} \overset{(d)}{=} (Z_x(u))_{\in \mathbb{R}^d}$$

with Z_x *a non degenerate random field.*

- ☛ Lass of order $h(x)$ at point $x \Longleftrightarrow$ Laos at x for $E(x) = Id/h(x)$.
- ☛ Operator scaling for $E \Longrightarrow$ Laos at 0 for E.
- ☛ Z_x operator scaling of order $E(x)$.

For any x, consider a matrix $E(x) \in \mathcal{M}_d(\mathbb{R})$. Assume that

- $\forall x \in \mathbb{R}^d$, $\min_{\lambda \in \mathrm{Sp} E(x)} \Re(\lambda) > 1$,
- $\psi_x : \mathbb{R}^d \backslash \{0\} \to (0, +\infty)$ continuous $E(x_0)^t$-homogeneous.

Then,

$$X_\alpha(x) = \Re \int_{\mathbb{R}^d} \left(\mathrm{e}^{ix\cdot\xi} - 1\right) \psi_x(\xi)^{-1-\mathrm{tr}(E(x))/\alpha} W_\alpha(d\xi)$$

is well-defined. We also assume some locally Lipschitz conditions on E and ψ and

- $E(y)E(w) = E(w)E(y)$ for y, w in a neighborhood of x.

Theorem 6 (Biermé, L., Scheffler, 2011)

X_α is laos for $E(x)$ at point x with $Z_x = X_{\alpha,\psi_x}$ a harmonizable operator scaling random field for $E(x)$ (defined with ψ_x).

☛ **Example** $E(x) = E_0/h(x)$ and $\psi_x = \psi^{h(x)}$ (this includes mutifractional Brownian motions)

Jordan decomposition of $E(x)$

$$E(x) = P(x)^{-1} \begin{pmatrix} J_1(x) & 0 & \dots & 0 \\ 0 & J_2(x) & 0 & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & J_{p_x}(x) \end{pmatrix} P(x).$$

- $J_k(x)$ associated with $a_k(x)$ the real part of the eigenvalue $\lambda_k(x)$.
- Let $(e_1, \dots, e_d)$ be the canonical basis of $\mathbb{R}^d$. For all $j = 1, \dots, p_x$, let

$$W_j(x) = \text{span}\left(P(x)^{-1} e_k \ ; \ \sum_{i=1}^{j-1} d_i + 1 \leq k \leq \sum_{i=1}^{j} d_i \right).$$

where $d_k = \text{size}(J_k(x))$.

Theorem 7 (Biermé, L., Scheffler, 2011)

- *If $\theta \in W_j(x) \cap \mathbb{S}^{d-1}$, a.s.*

$$H_{X_\alpha}(x,\theta) = \frac{1}{a_j(x)}$$

- *Moreover, a.s.*

$$H_{X_\alpha}(x) = \frac{1}{\max_{1 \le j \le d} a_j(x)}$$

☛ **Example 1** Let $E = E_0/h(x)$. Let $W_1, \ldots, W_p$ be the subspaces associated to the Jordan's decomposition of E_0 (resp. associated with $a_j^0 = \Re(\lambda_j^0)$).

If $u \in W_j \cap S^{d-1}$, for any $x \in \mathbb{R}^d$, a.s.

$$H_{X_\alpha}(x,u) = \frac{h(x)}{a_j^0}.$$

Moreover, for any $x \in \mathbb{R}^d$, a.s.

$$H_{X_\alpha}(x) = \frac{h(x)}{\max_{1 \le j \le d} a_j^{(0)}}.$$

☛ **Example 2** Consider $H_1, \ldots, H_d$ some locally Lipschitz functions and

$$E = P^{-1}\text{diag}\,(1/H_1, \ldots, 1/H_d)\, P.$$

Let $1 \leq j \leq d$ and set $f_j = P^{-1}e_j$. Then, for any $x \in \mathbb{R}^d$,

$$H_{X_\alpha}\left(x, \frac{f_j}{\|f_j\|}\right) = H_j(x) \quad \text{almost surely.}$$

☛ **Example 3** Let

$$E(x) = a(x)\begin{pmatrix} \cos(\theta(x)) & \sin(\theta(x)) \\ -\sin(\theta(x)) & \cos(\theta(x)) \end{pmatrix}$$

where a and θ are locally Lipschitz functions. Then, for every $x \in \mathbb{R}^d$ and $u \in \mathbb{S}^{d-1}$,

$$H_{X_\alpha}(x, u) = \frac{1}{a(x)\cos\left(\theta(x)\right)}.$$