Self-similar Sets and Martin Boundaries

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Outline of talk

1. Martin boundary
2. Self-similar set K and symbolic representation
3. Augmented rooted tree and hyperbolicity
4. Homeomorphism of K and the Martin boundary:
    a. Denker-Sato type random walk
    b. Transversal random walk
    c. Reversible random walk

Classical theory

Recall for bounded domain with smooth boundary, the solution of

$$\begin{cases} \triangle u = 0 & \text{on} \;\; \Omega \\ u = f & \text{on} \;\; \partial\Omega \end{cases}$$

is $u(x) = \int_{\partial\Omega} K(x,y) f(y) dy$ (harmonic function) where K is the Poisson kernel:

$$K(x,y) = \partial G(x,y)/\partial\nu, \qquad x \in \Omega, \;\; y \in \partial\Omega$$

G is the Green function , ν is the normal direction of Ω at y.

- For non-smooth boundary, the formula may not hold.

Martin (1941) enlarged the boundary $\partial\Omega$ by adding in abstract points to obtain a representation for +ve harmonic functions:

Martin kernel

$$K(x,y) = \frac{G(x,y)}{G(x_0,y)}, \qquad x,\ y \in \Omega$$

• With the weak topo. generated by $\{K(x,\cdot) : x \in \Omega\}$ on Ω, the compactification is $\overline{\Omega}_M$. The Martin boundary is defined as

$$\partial\Omega_M := \overline{\Omega}_M \setminus \Omega.$$

• $K(x,\cdot)$ is extended continuously to $\overline{\Omega}_M$. $\exists\ \partial\Omega_{\min} \subseteq \partial\Omega_M$ and μ supported by $\partial\Omega_{\min}$ $\ni$ for any +ve harmonic function u on Ω,

$$u(x) = \int_{\partial\Omega_{\min}} K(x,y) f(y) d\mu(y)$$

Doob (1959) carried the whole theory to the (transient) Markov chain $\{Z_n\}_{n=0}^{\infty}$ with a countable state space X. Let

$$p(x,y) = \mathbb{P}(Z_{n+1} = y / Z_n = x).$$

Green function: $g(x,y) = \sum_{n=0}^{\infty} p_n(x,y) < \infty$

Martin kernel: $k(x,y) = \dfrac{g(x,y)}{g(x_0,y)}$

Martin metric: $\theta(x,y) = \sum_{z \in X} a_z |k(x,z) - k(y,z)|$.

Martin boundary: $\mathcal{M} = \overline{X} \setminus X$ where $\overline{X}$ is the completion.

- The integral representation for harmonic functions on X ($Ph = h$) holds similarly.

- There is extensive study for X equipped with a group structure (Woess' book)

Self-similar set

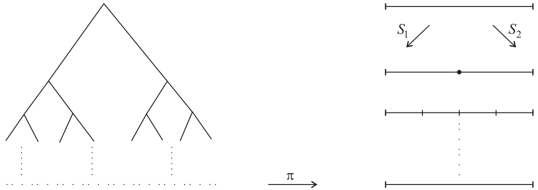
Let $\{S_j\}_{j=1}^N$ be an IFS of contractive similitudes (assume equal contraction $0 < r < 1$), and let

$$K = \bigcup_{j=1}^{N} S_j K \ .$$

Let $\Sigma = \{1, \cdots, N\}$, $\Sigma^* = \bigcup_{n=0}^{\infty} \Sigma^n$ (with $\Sigma^0 = \{\vartheta\}$).

• Symbolic representation: $x \in K$ can be represented by a sequence of finite words in Σ^*.

Example : $S_1(x) = \frac{1}{2}x, \; S_2(x) = \frac{1}{2}(x+1)$



Consider (Σ^∞, d) with $d(x,y) = r^{\min\{k: i_k \neq j_k\}}$, and let

$$\pi : (\Sigma^\infty, d) \to K \quad \text{by} \quad \pi(x) = \lim_{n\to\infty} K_{i_1 \cdots i_n} \; .$$

- (Σ^∞, d) is totally disconnected (Cantor type set), but $\Sigma^\infty / \sim_\pi$ is homeomorphic to K.

Question : Can we define Markov chains on Σ^* and identify K with the Martin boundary $\mathcal{M}$, and to make use of the induced harmonic structure on K ?

First studied by Denker & Sato (2001) on SG.

- An observation: For the IFS $\{S_j\}_{j=1}^N$ associated with probability weight $\{w_j\}_{j=1}^N$ (the usual way we consider iteration), the transition probability on Σ^* is

$$p(x, xj) = w_j.$$

Then $\mathcal{M}$ is homeomorphic to a Cantor set !!

- We need more structure on the tree Σ^*

Augmented rooted tree

Assume $\{S_j\}_{j=0}^N$ satisfies the OSC, we introduce a graph structure (Σ^*, E) ($x \sim y$ means $(x,y) \in E$) with $E = E_v \cup E_h$

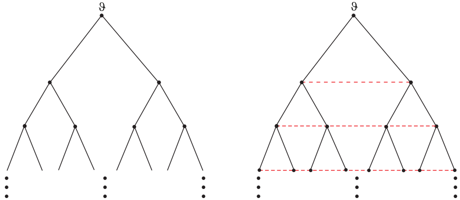
E_v is set (x,y) of vertical edges of the tree Σ^*;

E_h is the set of horizontal edges: $x, y \in \Sigma_n$,

$$x \sim y \ (\textit{neighbor}) \quad \textit{iff} \quad K_x \cap K_y \neq \emptyset.$$

Remark. (Kamanovich (2003) first introduced this on SG.)

Example : $S_1(x) = \frac{1}{2}x,\ S_2(x) = \frac{1}{2}(x+1)$



Theorem 1 (Wang & L (2008, Ind. U. Math. J.)). Assume $\{S_j\}_{j=0}^N$ satisfies the OSC, then

(i) (Σ^*, E) is a hyperbolic graph;

(ii) the hyperbolic boundary $\partial_H(\Sigma^*)$ is homeomorphic to K.

- Hyperbolic graph (Gromov) : $\exists\ \delta \geq 0$ such that all geodesic triangle are δ-thin, i.e., given a point on the triangle, the distance to the other two sides is less than δ.

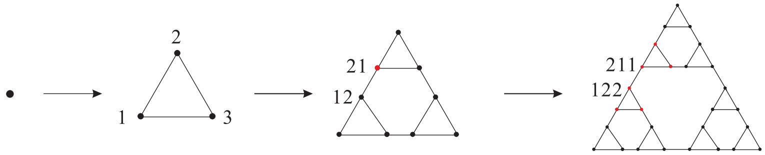
- For augmented rooted tree, this is equivalent to: for any $x, y \in \Sigma^*$, the horizontal segment of the geodesic is uniformly bounded

- We make use of this graph structure, and consider three types of Markov chains on (Σ^*, E)

Three types of Markov chains on Σ^*

Type I. Denker & Sato (2001): $\{Z_n\}_{n=0}^{\infty}$ on Σ_* of the SG.

At each point, the chain moves to the next level on its descends and the descends of its conjugates (neighbors with different parents, denote by $\asymp$) with equal weight. Then $K \approx \mathcal{M}$, and the hitting distribution is the Hausdorff measure on K.



This was extended to post critically finite (p.c.f) self-similar sets (Ju-Wang-L (2011,TAMS))

Theorem 2 (Wang & L) Suppose $\{S_j\}_{j=1}^n$ satisfies the OSC. Then for any DS-type Markov chain on the augmented tree of Σ^*, $K \approx \mathcal{M} = \mathcal{M}_{\min}$.

DS-type Markov chain: $x, y, z \in \Sigma^n$, $n \geq 1$:

(A1) (descendant) $p(x, xj) > 0$;

(A2) (conjugacy) $x \asymp y \Rightarrow p(x, yj) > 0$ for any $yj \asymp xi$;

(A3) (range) $p(x, zj) > 0 \Rightarrow$ either $x = z$ or $x \sim z$;

(A4) (uniformity) $\exists C > 0 \ni \forall\, x \in \Sigma^\infty$,

$$\frac{g(\vartheta, u)}{g(\vartheta, v)} \leq C,$$

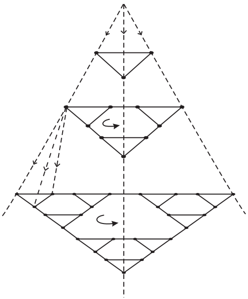
where $u, v \in \mathcal{A}(x)$ (ancestor set of x) and $|u| = |v|$.

• In some special cases we can show that the hitting distribution is a Hausdorff measure

Question: Can the DS-type Markov chain be represented as a graph directed system of IFS ?

Type II: A traversal walk on Σ_* of the SG (Ngai & L).

$$p(x,y) = \begin{cases} 1/3, & \text{if } x, y \in \Sigma^n \setminus V^n,\ x \sim y; \\ 1/3, & \text{if } x \in V^n,\ v = xi,\ i = 1,2,3; \\ 0, & \text{otherwise.} \end{cases}$$



Theorem 3. $\lim_{n\to\infty} Z_n = Z_\infty$ *P_ϑ-a.s. where Z_∞ is a $\{\dot{1}, \dot{2}, \dot{3}\}$-valued r.v. Moreover,*

$$\mathcal{M} \approx SG \quad \textit{and} \quad \mathcal{M}_{\min} = \{\dot{1}, \dot{2}, \dot{3}\}.$$

Recall from the general theory, a bounded P-harmonic function has the representation

$$h(x) = \int_{\mathcal{M}_{min}} K(x,\xi)\phi(\xi)d\nu_\vartheta(\xi)$$

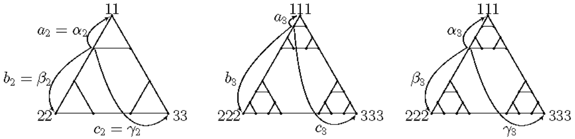
for some bounded function ϕ on $\mathcal{M}_{min}$.

Theorem 4. *The class of P-harmonic functions is 3-dimensional. There is an induced "$1/5 - 2/5$ rule" which yields a natural identification with the canonical ones (Kigami.)*

The main proof is on $\mathcal{M} \approx SG$.

Hitting probability on Σ^n: $\boldsymbol{\rho}(x) = [\rho_1(x), \rho_2(x), \rho_3(x)]$, the probability of reaching the three vertices from $x \in \Sigma^n$.

Let $\alpha_n,\ \beta_n,\ \gamma_n$ and $a_n,\ b_n,\ c_n$ as indicated, they can be calculated inductively.



and define

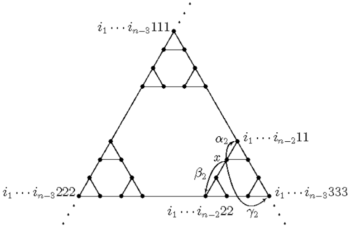
$$A_n^{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ \alpha_n & \beta_n & \gamma_n \\ \alpha_n & \gamma_n & \beta_n \end{bmatrix}, \quad A_n^{(2)} = \begin{bmatrix} \beta_n & \alpha_n & \gamma_n \\ 0 & 1 & 0 \\ \gamma_n & \alpha_n & \beta_n \end{bmatrix}, \quad A_n^{(3)} = \begin{bmatrix} \beta_n & \gamma_n & \alpha_n \\ \gamma_n & \beta_n & \alpha_n \\ 0 & 0 & 1 \end{bmatrix}.$$

It follows that

$$A_n^{(i)} = \begin{bmatrix} \rho(i1^{n-1}) \\ \rho(i2^{n-1}) \\ \rho(i3^{n-1}) \end{bmatrix}, \quad i = 1, 2, 3.$$

Hence for $x = i_1 \cdots i_n$,

$$\rho(x) = \mathbf{e}_{i_n} A_2^{(i_{n-1})} \begin{bmatrix} \rho(i_1 \cdots i_{n-2}11) \\ \rho(i_1 \cdots i_{n-2}22) \\ \rho(i_1 \cdots i_{n-2}33) \end{bmatrix} = \mathbf{e}_{i_n} A_2^{(i_{n-1})} \cdots A_{n-1}^{(i_2)} A_n^{(i_1)}.$$



- By using the maximum range consideration of matrices (the maximum difference of two columns) (Hajnal (1958)), we show that $A_n^{(i)}, i = 1, 2, 3$ converges to

$$A^{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ 2/5 & 2/5 & 1/5 \\ 2/5 & 1/5 & 2/5 \end{bmatrix}, \quad A^{(2)} = \begin{bmatrix} 2/5 & 2/5 & 1/5 \\ 0 & 1 & 0 \\ 1/5 & 2/5 & 2/5 \end{bmatrix}, \quad A^{(3)} = \begin{bmatrix} 2/5 & 1/5 & 2/5 \\ 1/5 & 2/5 & 2/5 \\ 0 & 0 & 1 \end{bmatrix},$$

and for $\mathbf{x} = i_1 i_2 \cdots \in \Sigma_\infty$

$$\lim_{n\to\infty} A_2^{(i_{n-1})} \cdots A_{n-1}^{(i_2)} A_n^{(i_1)} \qquad \text{exists.}$$

So is $\lim_{n\to\infty} \rho(\mathbf{x}|_n)$.

For the Green function and the Martin kernel, we have

$$K(x,y) = \frac{G(x,y)}{G(\vartheta,y)} = \frac{\sum_{i=1}^{3} b_i^{m,n-1}(x) G(i^{n-1},y)}{(1/3)\sum_{i=1}^{3} G(i^{n-1},y)} \ .$$

where

$$b_j^{m,n-1}(x) := \begin{cases} \sum_{i=1}^{3} \rho_{x,i^m} \ \rho_{i^m,j^{n-1}}, & \text{if } x \in \Sigma^m \setminus V^m \\ \rho_{x,j^{n-1}}, & \text{if } x \in V^m. \end{cases}$$

By using this we can identify $\mathcal{M}$ and prove the homeomorphism with the SG

• A well known open question : Does a self-similar set (connected and with OSC) admit a Laplacian.

• The question is known only for some highly symmetric p.c.f. self-similar sets (Kigami, Lindstrom) (and some extensions), and also the Sierpinski carpet (Barlow & Bass).

• Theorem 5 says that the resulting harmonic structure on the SG coincide with the well-known one.

• Our method can also be applied to the Hata tree (non-symmetric p.c.f. self-similar set)

Question. Can the method be applied to the more general p.c.f similar sets?

Type III Reversible Markov chain on (Σ^*, E) (Wong & L)

$$p(x,y) = \begin{cases} \frac{c(x,y)}{m(x)}, & x \sim y \\ 0, & \text{otherwise.} \end{cases}$$

where $c(x,y) = c(y,x)$, $0 < M_1 \le c(x,y) \le M_2$ and $m(x) = \sum_{y:y\sim x} c(x,y)$.

Example: $\{Z_n\}_{n=0}^{\infty}$ is the nearest neighborhood r.w. $(c(x,y) = 1)$.

Theorem 5. *Let p be as above, then $K \approx \partial_H(\Sigma_*) \approx \mathcal{M} = \mathcal{M}_{\min}$*

- Reversibility together with a theorem of Ancona (1987) on hyperbolic graph imply $\partial_H(\Sigma^*) \approx \mathcal{M}$.
- $K \approx \partial_H(\Sigma^*)$ is due to Wang & L (2008).

Recall the Dirichlet energy of f on $\mathbb{D}$ is $\mathcal{E}_{\mathbb{D}}[f] = \int_{\mathbb{D}} |\nabla f|^2$. The induced energy form of u on the unit circle $\partial\mathbb{D}$ is (Douglas integral)

$$\mathcal{E}_{\partial\mathbb{D}}[u] = \mathcal{E}_{\mathbb{D}}[Hu] = \frac{\pi}{4}\int_0^{2\pi}\int_0^{2\pi} (u(\theta) - u(\eta))^2 / \sin^2\left(\frac{\theta - \eta}{2}\right) d\theta d\eta$$

We make use the above theorem to consider a similar induce energy form on the Martin boundary. It was motivated by a similar consideration by Kigami (preprint) on the tree Σ^*.

We define a graph energy on (Σ^*, E):

$$\mathcal{E}[f] = \frac{1}{2} \sum_{x \sim y} c(x, y)(f(x) - f(y))^2.$$

Proposition 6. *The graph energy induces a closed, non-negative definite, symmetric bilinear form* $(\mathcal{E}_K, \mathcal{D}_K)$

$$\mathcal{E}_K(u, v) = \mathcal{E}(Hu, Hv)$$

where Hu *on* Σ^* *is the Poisson integral of* u *on* K*.*

- The above form is called a Dirichlet form if $\mathcal{D}_K$ is dense in $L^2(K, \nu_\vartheta)$, and satisfies:

$$u \in \mathcal{D}_K \Rightarrow (0 \vee u) \wedge 1 \in \mathcal{D}_K, \text{ and } \mathcal{E}[v] \leq \mathcal{E}[u].$$

$\{Z_n\}_{n=0}^{\infty}$ is said to satisfy a uniform drift condition if

$$p_1 := \inf_x \mathbb{P}_x\{|Z_1| = |x| + 1\} > q_1 := \sup_x \mathbb{P}_x\{|Z_1| = |x| - 1\}$$

Theorem 7. *Suppose* $\dim_H K < 2$*, and the random walk is reversible and satisfies the above condition. Then* $\mathcal{E}_K$ *is a non-local Dirichlet form on* $L^2(K, \nu_\vartheta)$*, and*

$$\mathcal{E}_K[u] = \int_K \int_K (u(x) - u(y)^2 J(x, y) d\nu_\vartheta(x) d\nu_\vartheta(y).$$

- The main proof is to show for $\dim_H K < 2$, then $Lip(K, 1) \subset \mathcal{D}_k$ (using the above UDC), hence $\mathcal{D}_K$ is dense in $L^2(K, \nu_\vartheta)$.
- Then integral expression is by Silverstein (1974).

There are many unanswered questions:

Q1. How to eliminate $\dim_H K < 2$?

Q2. Is the uniform drift condition necessary?

Q3. What is the property of hitting distribution (harmonic measure) of ν_ϑ and ν_x?

Q4. The regularity of $(\mathcal{E}_K, \mathcal{D}_K)$?

Q5. The Dirichlet form can be used to define a jump process. Study this process and the heat kernel.