Multifractal analysis of Banach valued Birkhoff ergodic averages for interval maps with infinitely many branches

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Outline

General problems and classical results

I. General problem

(X, T) a dynamical system

- X metric space (compact).
- T continuous (piecewise continuous).

$\mathbb{B}$ a real Banach space, $\mathbb{B}^*$ its dual space
$\Phi : X \to \mathbb{B}^*$ (piecewise continuous).

- Birkhoff ergodic averages of Φ :

$$A_n\Phi(x) := \frac{1}{n}\sum_{j=0}^{n-1}\Phi(T^j x) \qquad (n \geq 1).$$

- Level sets :

$$X_\Phi(\alpha) := \left\{x : \lim_{n\to\infty} A_n\Phi(x) \stackrel{\mathbb{B}}{=} \alpha\right\}, \quad \alpha \in \mathbb{B}^*.$$

Question : What are the sizes of $X_\Phi(\alpha)$?

- Multifractal spectrum : $f(\alpha) := \dim_H X_\Phi(\alpha)$.

II. A classical example

- $X = [0,1],\ Tx = 2x \bmod 1,\ \Phi(x) = 1_{[\frac{1}{2},1]}$.
  - Each number $x \in [0,1]$, $x = \sum_{n=1}^{\infty} \frac{x_n}{2^n}$ $(x_n = 0,1)$. Then

$$\lim_{n\to\infty} A_n\Phi(x) = \lim_{n\to\infty} \frac{x_1 + \cdots + x_n}{n}$$

    is the frequency of the digit 1 in the development of x in base 2.

  - **E. Borel 1909 :**

$$\lim_{n\to\infty} A_n\Phi(x) = \frac{1}{2} \quad \mathcal{L} - a.e.$$

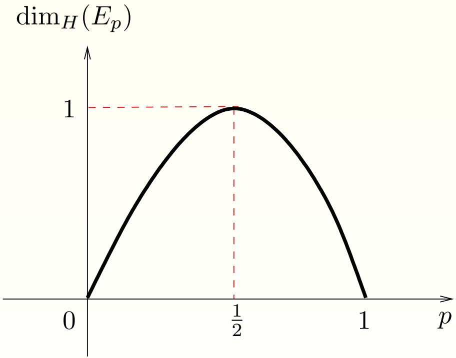
  - **A. Besicovitch, 1934 ; H. G. Eggleston, 1949 :**

$$\dim_H(E_p) = \frac{-p\log p - (1-p)\log(1-p)}{\log 2}.$$

    where

$$E_p := \Big\{x \in [0,1] : \lim_{n\to\infty} A_n\Phi(x) = p\Big\}.$$

Spectrum of Besicovitch-Eggleston sets



III. Interval maps with finite branches

Let $\mathcal{M}$ be the set of T-invariant measures.

- $h(\mu)$: the entropy of μ
- $\lambda(\mu) = \int \log |T'| d\mu$: the Lyapunov exponent with respect to μ.

In general, we have

$$\dim_H(X_\Phi(\alpha)) = \sup\left\{\frac{h(\mu)}{\lambda(\mu)} : \ \mu \in \mathcal{M}, \int \Phi d\mu = \alpha\right\}.$$

IV. An example of Banach valued functions

- $X = \{0,1\}^{\mathbb{N}}$, T be the left shift, $\Phi(x) = (1_{[0]}, 1_{[0^2]}, 1_{[0^3]}, \dots)$
- $\alpha = (a_1, a_2, \cdots) \in \ell^\infty$: a sequence of non-negative numbers.
- $X_\Phi(\alpha)$ corresponds to the set of numbers $x \in [0,1]$ such that in the development of x in base 2, the frequency of the block 0^k is a_k for all $k \geq 1$.

Theorem (Fan-L-Peyrière 2008)

The set $X_\Phi(\alpha)$ is non-empty iff

$$1 = a_0 \geq a_1 \geq a_2 \geq \dots; \quad a_i - 2a_{i+1} + a_{i+2} \geq 0 \quad (i \geq 0). \tag{1}$$

If the condition (1) is fulfilled,(write $h(x) = -x \log_2 x$)

$$\begin{aligned} \dim_H X_\Phi(\alpha) &= -h(1-a_1) + \sum_{j=0}^{\infty} h(a_j - 2a_{j+1} + a_{j+2}) \\ &= \sup\left\{ \frac{h(\mu)}{\lambda(\mu)} : \ \mu \in \mathcal{M}, \int \Phi d\mu = \alpha \right\}. \end{aligned}$$

V. Gauss dynamical system and continued fraction

- Gauss dynamical system $T : [0,1] \to [0,1]$:

$$T(0) := 0, \quad T(x) := \frac{1}{x} \pmod 1, \ \forall x \in (0,1]$$

Fraction continue : $x \in [0,1]$,

$$x = \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \cfrac{1}{a_3(x) + \ddots}}}$$

where $a_1(x) = \lfloor 1/x \rfloor$, et $a_n(x) = a_1(T^{n-1}(x))$, $\forall n \geq 2$.

- $\Phi(x) = (1_{[j]})_{j \geq 1} \circ a_1(x), \quad \alpha = (p_1, p_2, \dots)$.

$$X_\Phi(\alpha) = \left\{ x : \frac{1}{n} \sharp\{k \leq n : a_k(x) = j\} \to p_j \ \forall j \geq 1 \right\}.$$

It is the set of numbers with the frequency of the digit j equal to p_j.

VI. Besicovitch-Eggleston set in continued fractions

Denote $\mathcal{N}$ the set of measures μ :

- μ ergodic
- $\mu([j]) = p_j$
- $\int \log x d\mu < \infty \quad (h(\mu) < \infty)$.

Theorem (Fan-L-Ma 2010)

With the convention : $\sup \emptyset = 0$, we have

$$\dim_H X_\phi(\alpha) = \max \left\{ \frac{1}{2}, \quad \sup \left\{ \frac{h(\mu)}{2 \int \log x d\mu} : \mu \in \mathcal{N} \right\} \right\}$$

VII. Development in the last years

Symbolic dynamical systems on an alphabet of finite symbols and related systems : **Barreira, Barral, Climenhaga, Fan, Feng, Hofbauer, Jordan, Lau, Liao, Ma, Mensi, Olivier, Olsen, Pesin, Peyrière, Reeve, Pfister, Saussol, Schmeling, Shu, Simon, Sullivan, Taken, Tempelman, Thompson, Verbytzky, Weiss, Wen, Wu, ...**

Infinite symbols and continued fractions : **Fan, Iommi, Jaerisch, Jordan, Kesseböhmer, Liao, Ma, Mauldin, Munday, Pollicott, Rams, Stratmann, Urbański, Wang, Weiss, Wu, ...**

Interval maps with infinitely many branches

I. Setting

Dynamical system :

- $\{I_i\}_{i=1}^{\infty}$ be a countable collection of disjoint intervals in $[0,1]$.
- $T_i : \overline{I_i} \to [0,1]$ be a bijective $C^{1+\eta}$ map such that $|T_i'(x)| \geq \xi > 1$.

Define the map $T : \cup I_i \to [0,1]$ by

$$T(x) = T_i(x) \text{ if } x \in I_i$$

and let

$$\Lambda = \cap_{n=0}^{\infty} T^{-n}([0,1]).$$

$\rightarrow$: Dynamical system (Λ, T).

Functions : $\Phi = (\phi_1, \phi_2, \dots)$ with $\phi_i : \Lambda \to \mathbb{R}$ continuous.

Sets : For $\alpha = (\alpha_1, \alpha_2, \dots) \in \mathbb{R}^{\mathbb{N}}$,

$$X_\Phi(\alpha) := \left\{ x : \lim_{n\to\infty} A_n\phi_i(x) = \alpha_i \text{ for all } i \in \mathbb{N} \right\}.$$

II. Hypotheses and notation

For $\phi : \Lambda \to \mathbb{R}$, define its n-th variation by

$$\mathrm{var}_n(\phi) = \sup\left\{|\phi(x) - \phi(y)| : x, y \in T_{\omega_1}^{-1} \circ \cdots \circ T_{\omega_n}^{-1}([0,1]),\ (\omega_1 \cdots \omega_n) \in \mathbb{N}^n\right\}.$$

We say that ϕ **has variations uniformly converging to 0** if

$$\mathrm{var}_1(\psi) < \infty \text{ and } \lim_{n\to 0} \mathrm{var}_n(\psi) = 0.$$

Hypothesis : $\log |T'|$ and ϕ_i $(i \in \mathbb{N})$ all have variations uniformly converging to 0.

Denote by $\mathcal{M}$ the set of invariant measures. Let

$$Z_0 = \left\{\alpha \in \mathbb{R}^{\mathbb{N}} : \exists \mu \in \mathcal{M}, \forall i \in \mathbb{N}, \int \phi_i \mathrm{d}\mu = \alpha_i\right\}.$$

and denote by Z the **closure** of Z_0 in the pointwise limit topology.

Let

$$s_\infty = \inf\left\{s : \sum_{i\in\mathbb{N}} \mathrm{diam}(I_i)^s < \infty\right\}.$$

III. Theorem

Theorem (Fan-Jordan-L-Rams 2011 preprint)

For $\alpha \notin Z$, $X_\Phi(\alpha) = \emptyset$. For $\alpha \in Z$, the dimension $\dim_H X_\Phi(\alpha)$ is given by

$$\lim_{\varepsilon\to 0}\lim_{k\to\infty}\sup_{\mu\in\mathcal{M}}\left\{\frac{h(\mu)}{\lambda(\mu)} : \left|\int \phi_i \mathrm{d}\mu - \alpha_i\right| < \varepsilon\ \forall i \le k,\ h(\mu) < \infty\right\}.$$

Furthermore, if all ϕ_i are **bounded**, then
for all $\alpha \in Z \setminus Z_0$ we have $\dim_H X_\Phi(\alpha) = s_\infty$, and for $\alpha \in Z_0$,

$$\dim_H X_\Phi(\alpha) = \max\left\{s_\infty,\quad \sup_{\mu\in\mathcal{M}}\left\{\frac{h(\mu)}{\lambda(\mu)} :\ \int \Phi d\mu = \alpha,\ h(\mu) < \infty\ \right\}\right\}.$$

IV. Examples

1. Continued fractions : for $\alpha \in [0,1]$, let

$$\Lambda_\alpha = \left\{ x \in [0,1]\backslash\mathbb{Q} : \lim_{n\to\infty} \frac{\frac{1}{a_1(x)} + \frac{1}{a_2(x)} + \cdots + \frac{1}{a_n(x)}}{n} = \alpha \right\}$$

then we have

$$\dim \Lambda_\alpha = \max \left\{ \frac{1}{2}, \sup_{\mu \in \mathcal{M}} \left\{ \frac{h(\mu)}{\lambda(\mu)} : \int \phi \mathrm{d}\mu = \alpha, \ h(\mu) < \infty \right\} \right\}.$$

2. A new phenomenon : Let $\Phi(x) = 1_{I_1}$. Suppose

$$|I_n| \approx \frac{1}{n^2(\log n)^4}.$$

The spectrum of $[0,1] \ni \alpha \mapsto \dim_H X_\Phi(\alpha)$ has two **flat** parts in both end of 0 and 1.