AN ALGORITHM TO COMPUTE THE CENTERED HAUSDORFF MEASURE OF SELF-SIMILAR SETS

Fractals and Related Fields II

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[1] Joint work with M. Morán

Self-similar sets

- $\Psi = \{f_1, f_2, ..., f_m\}$ a system of contracting similitudes of $\mathbb{R}^n$
  Self-similar set E: The unique ($\neq \emptyset$) compact set satisfying

$$E = \bigcup_{i=1}^{m} f_i(E)$$

- Open set condition **(OSC)** :

$$\exists\ \mathcal{O} \subset \mathbb{R}^n \text{ open} : \begin{cases} f_i(\mathcal{O}) \subset \mathcal{O} \ \ i = 1, ..., m \\ f_i(\mathcal{O}) \cap f_j(\mathcal{O}) = \emptyset \text{ for } i \neq j. \end{cases}$$

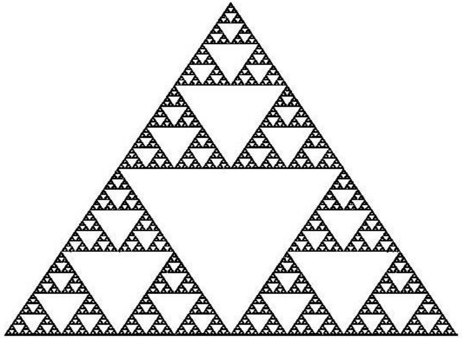
- Ψ satisfies the strong separation condition, **(SSC)**, if
  $f_i(E) \cap f_j(E) = \emptyset\ \forall i \neq j$.

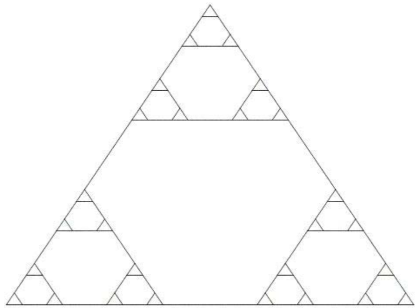
Geometric interpretation: OSC-SSC

OSC$\equiv$ If the pieces of the self-similar set *(basic cylinder sets)*, do intersect, this intersection is "small" :

$$H^s(f_i(E) \cap f_j(E)) = 0 \ \forall i \neq j,$$

where $s = \dim E$ y $H^s(E)$ is the s-dimensional Hausdorff measure.
SSC$\equiv$ The pieces of the self-similar set *(basic cylinder sets)* are disjoint



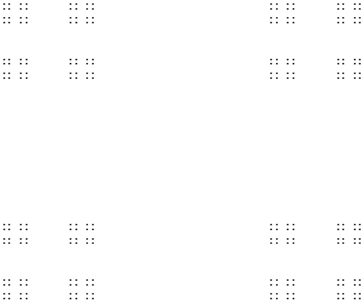


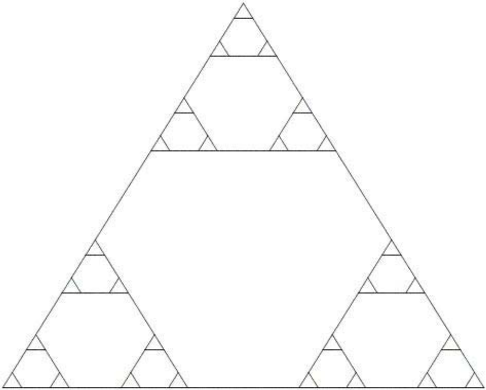
$$\textbf{OSC} \implies \begin{cases} \dim_H E = s \begin{cases} s \text{ is the solution } \sum_{i=1}^{m} r_i^s = 1, \\ r_i \text{ is the contraction ratio of } f_i \end{cases} \\ 0 < H^s(E) < \infty. \end{cases}$$

SSC $\implies$ invariant set E is totally disconnected (Euclidean metric)

Two self-similar sets of the same dimension ($\sum_{i=1}^{m} r_i^s = 1$)

$$\dim_H C(1/4) = 1 = \dim_H S(1/3)$$





Two equivalent measures: $2^{-s}C^s(E) \leq H^s(E) \leq C^s(E)$

Hausdorff measure

$$H^s(E) = \lim_{\varepsilon \to 0} \left[\inf \left\{ \sum_{i=1}^{\infty} |U_i|^s \ : \ |U_i| \leq \varepsilon, \ E \subset \cup U_i \right\} \right]$$

Hausdorff centered measure

$$C_0^s(E) = \lim_{\varepsilon \to 0} \Big[\inf \Big\{ \sum_{i=1}^{\infty} |B_i|^s \ : |B_i| \leq \varepsilon, \ E \subset \cup B_i, \\ B_i \text{ closed balls centered in } E \Big\} \Big]$$

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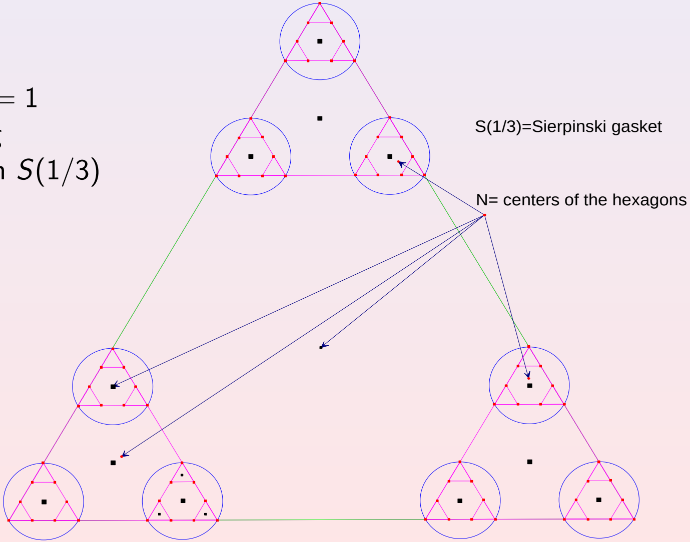
$C_0^s(E)$ not monotone !!!!

$$C^s(E) = \sup\{C_0^s(F) \ : \ F \subset E\} \ \text{(Measure)}$$

EXAMPLE: $C_0^s(S(1/3) \cup N) < C_0^s(S(1/3))$ [Tricot ('82)]

$s = dim_H(S(1/3) = 1$
Any other covering
by balls centered in $S(1/3)$
is less efficient!!

$N = \cup_{i=1}^{\infty} x_i$
$H^1(N) = 0$



Two Steps Measures:

If E is the self-similar set associated to the system $\Psi = \{f_1, f_2, ..., f_m\}$ satisfying the OSC with $\dim_H E = s$.

Theorem (LL. and Morán (´10))

If $F \subset E$ either closed or open. Then $C_0^s F = C^s F$.

Two Steps Measures:

Packing measure $P^s(E) = \inf \left\{ \sum_{i=1}^{\infty} P_0^s(F_i) \ : \ E \subset \bigcup_{i=1}^{\infty} F_i \right\}$

$$P_0^s(E) = \lim_{\varepsilon \to 0} \Big[\sup\{ \sum_{i=1}^{n} |B_i|^s \ : |B_i| \leq \varepsilon, B_i \cap B_j = \emptyset \text{ for } i \neq j, B_i \text{ closed balls centered in } E \} \Big]$$

[Feng (´98)]: If K is compact: $P_0^s(K) < \infty \implies P_0^s(K) = P^s(K)$

$C^s(E), P^s(E)$ two step definition $\begin{cases} C_0^s(E) \text{ not monotone} \\ P_0^s(E) \text{ not } \sigma\text{-subadditive} \end{cases}$

The Lemma is false for general compact sets

Two Steps Measures:

Example

$\exists\ S \subset \mathbb{R}^n$ compact set $: C^s S > C_0^s S.$

$$C^1(S(1/3) \cup N) = C^1(S(1/3) = C_0^1(S(1/3) > C_0^1(S(1/3) \cup N)$$

since, the previous theorem

$$C^1(S(1/3) = C_0^1(S(1/3)$$

and

$$C^1(N) = 0$$

$(dim_H(S(1/3) = 1)$

Program to compute $C^s(E)$

STEP 1: ¿Optimal coverings? $\rightarrow$ **Optimization of density functions**

STEP 2: **Discretization of the problem** to construct an algorithm that converges to $C^s(E)$

Program to compute $C^s(E)$

STEP 1: ¿Optimal coverings? $\rightarrow$ **Optimization of density functions**

TOOLS:
1. $C_0^s F = C^s F$ (LL. & Morán (´10))
2. Self-similar tiling principle (Morán (´04))
3. Invariance of the density functions

$$\Longrightarrow \; C^s E = \min\left\{ \frac{(2r)^s}{\mu(B(x,r))} : x \in E \text{ and } c \leq r \leq R \right\}$$

STEP 2: **Discretization of the problem** to construct an algorithm that converges to $C^s(E)$

Program to compute $C^s(E)$: **STEP 1**

[Ayer and Stricharz (´99)]: *Exact Hausdorff measure and intervals of maximum density for Cantor sets*

[Morán (´04)]:*Computability of the Hausdorff and packing measures on self-similar sets and the self-similar tiling principle*

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$$H^s(E) \to \frac{H^s(E \cap I)}{|I|^s} = d(I)$$

$I \subset [0, 1] \to$ interval that maximizes the density

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$$\textbf{OSC} \begin{cases} H^s(E) = \inf\{|A|^s/\mu A : A \text{ is a convex polytope}\} \\ H^s_{sph}(E) = \inf\{2r^s/\mu B(x,r) : x \in \mathbb{R}^n\} \\ P^s(E) = \sup\{(2r)^s/\mu B(x,r) : x \in E, B(x,r) \subset \mathcal{O}\}. \end{cases}$$

$\mu = \frac{H^s_{\lfloor E}}{H^s(E)} \to$ natural probability measure on E [$\mu(f_i(E)) = r_i^s$)].

Program to compute $C^s(E)$: **STEP 1**

Theorem (LL. and Morán (´10))

Let E be be the invariant set of the system Ψ satisfying the open set with $\dim_H E = s$ and $|E| = R$ and let μ be the Hausdorff normalized measure on E . Then

$$C^s E = \inf\left\{\frac{(2r)^s}{\mu(B(x,r))} : x \in E, r > 0\right\} =: D_C^{-1} \quad (1)$$

Moreover, if Ψ satisfies the SSC then

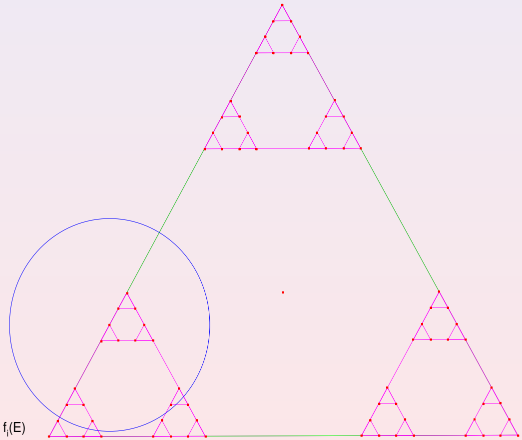
$$C^s E = \min\left\{\frac{(2r)^s}{\mu(B(x,r))} : x \in E \ \text{ and } c \leq r \leq R\right\}$$

where $c := \min_{i,j\in M, i\neq j} dist(f_i E, f_j E)$ and $R := |E|$.

Program to compute $C^s(E)$: **STEP 1**

$$\mu(f_i(E)) = r_i^s \mu(E)$$

$$\frac{(2r/r_i)^s}{\mu(f_i(B))} = \frac{(2r)^s}{\mu(B)}$$



Program to compute $C^s(E)$: **STEP 2**

Knowing that

$$C^s(E) = min\{f_s(x,d) = \frac{(2d)^s}{\mu(B(x,d))} : x \in E,\ d \in [c,1]\}$$

We construct:

1. a sequence $\{A_k\}$ of sets and a sequence $\{\mu_k\}$ of measures s.t.

$$spt(\mu_k) = A_k \qquad \mu_k \rightharpoonup \mu \qquad \text{and} \qquad E = \overline{\cup_{k=1}^{\infty} A_k}$$

2. with these sequences we construct another sequence $\{\widetilde{m_k}\}$

$$\widetilde{m_k} = f_k(\widetilde{x_k}, \widetilde{d_k}) = \min\{f_k(x_k, d_k) : d_k = ||x_k - y_k||, (x_k, y_k) \in A_k \times A_k\}$$

where $f_k(x_k, d_k) = \frac{(2d_k)^s}{\mu_k(B(x_k, d_k))}$

Theorem (LL. and Morán (´10/´11))

$\widetilde{m_k} \to C^s(E)$ *as* $k \to \infty$

$A_1=\{x_0, x_1, x_2\}$

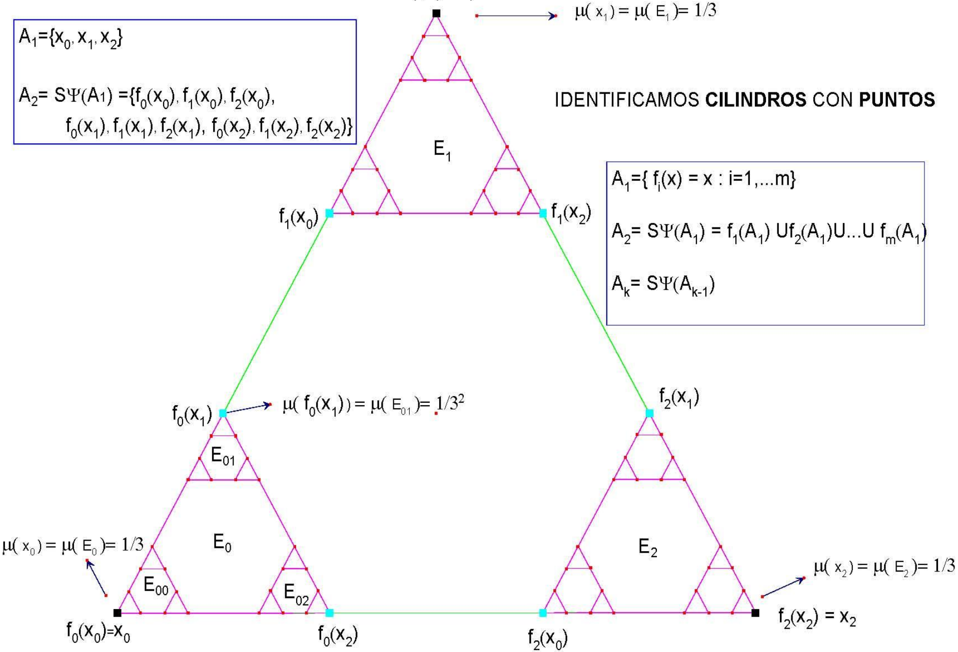
$A_2= S\Psi(A_1) =\{f_0(x_0), f_1(x_0), f_2(x_0), f_0(x_1), f_1(x_1), f_2(x_1), f_0(x_2), f_1(x_2), f_2(x_2)\}$

IDENTIFICAMOS **CILINDROS** CON **PUNTOS**

$A_1=\{ f_i(x) = x : i=1,...m\}$

$A_2= S\Psi(A_1) = f_1(A_1) \cup f_2(A_1) \cup ... \cup f_m(A_1)$

$A_k= S\Psi(A_{k-1})$



Examples: testing efficiency

λ-Cantor sets on the real line $K(\lambda) = \cup_{i=0}^{1} f_i(K(\lambda))$:

$$f_0(x) = \lambda x, \quad f_1(x) = 1 - \lambda + \lambda x, \quad x \in [0, 1]$$

[Zhu & Zhou (´02)]:

$$C^s(K(\lambda)) = 2^s(1-\lambda)^s \ \forall\ 0 < \lambda \leq \frac{1}{3} \qquad s = \frac{\log 2}{-\log \lambda}$$

$$K(\frac{1}{3}) - \textbf{Middle-third Cantor set:} \quad C^s(K(\frac{1}{3})) = \frac{4^{\frac{\log 2}{\log 3}}}{2} \simeq 1.199023$$

The algorithm finds the exact value of $C^s(K(\frac{1}{3}))$ already at the third iteration!!!: $\forall k \geq 2$ the two minimizing intervals selected, $[-\frac{1}{3}, 1]$ and $[0, \frac{4}{3}]$, verify

$$f_k(\frac{1}{3}, \frac{2}{3}) = f_k(\frac{2}{3}, \frac{2}{3}) = \frac{(2\frac{2}{3})^s}{1} = \frac{4^{\frac{\log 2}{\log 3}}}{2} = C^s(K(\frac{1}{3})).$$

Examples: **Middle-third Cantor set**

k	$\tilde{m}_k$	$(\tilde{x}_k, \tilde{y}_k)$	$\tilde{d}_k$	$B(\tilde{x}_k, \tilde{d}_k)$
0	1.54856	$(0,1)$ $(1,0)$	1	$[-1,1]$
1	1.03238	$(\frac{2}{3},\frac{1}{3})$ $(\frac{1}{3},\frac{2}{3})$	0.333333	$[\frac{1}{3},1]$ $[0,\frac{2}{3}]$
2	1.19902	$(\frac{2}{3},0)$ $(\frac{1}{3},1)$	0.666667	$[0,\frac{4}{3}]$ $[-\frac{1}{3},1]$
3	1.19902	$(\frac{2}{3},0)$ $(\frac{1}{3},1)$	0.666667	$[0,\frac{4}{3}]$ $[-\frac{1}{3},1]$
4	1.19902	$(\frac{2}{3},0)$ $(\frac{1}{3},1)$	0.666667	$[0,\frac{4}{3}]$ $[-\frac{1}{3},1]$
5	1.19902	$(\frac{2}{3},0)$ $(\frac{1}{3},1)$	0.666667	$[0,\frac{4}{3}]$ $[-\frac{1}{3},1]$

Examples: Conclusions

Testing the algorithm for the known cases

- λ-**Cantor sets on the real line**[Zhu & Zhou (´02)]
- (λ_1, λ_2)-**Symmetry Cantor sets on the real line**[Dai & Tian (´05)]
- **Cantor type sets on the plane**[Zhu & Zhou (´08)]

All of them have extra conditions $\Longrightarrow$ the contraction factors are quite small $\Longrightarrow$ the algorithm finds the exact value very quickly as

$$\mu(B(\tilde{x}_k, \tilde{d}_k)) = \mu_k(B(\tilde{x}_k, \tilde{d}_k)) = 1!!!!$$

(the optimal ball does not cut any cylinder set as it contains the whole set, so one can find a way to compute them "by hand")

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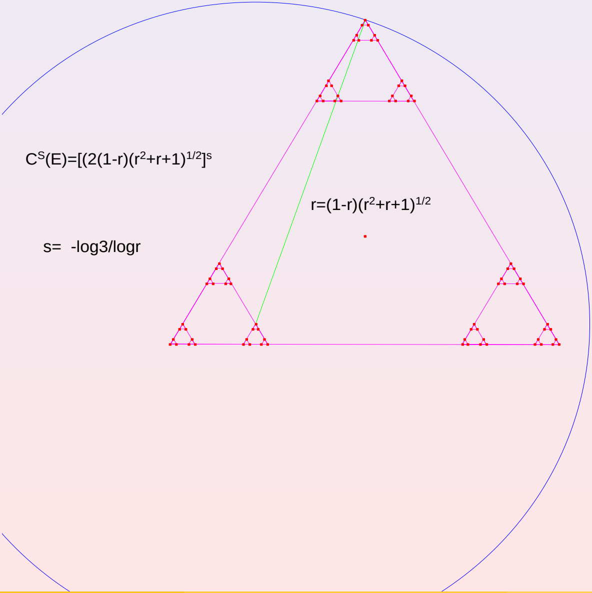
Examples: Sierpinski with $r < 0.25$



Examples: Self-similar sets with dimension ≥ 1

The optimal ball changes slightly: the center is fixed and the radius changes slightly

- $S(\frac{1}{3})$ **-Sierpinski gasket, (**$r = \frac{1}{3}$, $\dim_H = 1$ **)**

$$C^1(S(\frac{1}{3})) \simeq 1.54$$

- $C(\frac{1}{4})$**-Cantor set in the plane, (**$r = \frac{1}{4}$, $\dim_H = 1$**)**

$$C^1(S(\frac{1}{3})) \simeq 1.95$$

$$\{2.66667, 1.92296, 1.95814, 1.95542, 1.95306, 1.95388, 1.95417\}$$

- $C(\frac{4}{10})$**-Cantor set in the plane,** $r = \frac{4}{10}$,$\dim_H = \frac{-\log 4}{\log 0.4} \simeq 1.51294$

$$C^{\frac{-\log 4}{\log 0.4}}(S(\frac{1}{3})) \simeq 1.64$$

$$\{3.80522, 1.33333, 1.59995, 1.6425, 1.64928, 1.64807, 1.64972\}$$

$C^1(C(\frac{1}{4})) \simeq 1.95$