Applications of Gaussian Information plus Noise models to source localization

Ph. Loubaton
Joint work with W. Hachem, X. Mestre, J. Najim, P. Vallet

Random matrices and their applications
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1. Problem statement and overview of the results.
2. The case K fixed.
3. K may scale with M, N.
4. Some numerical results..

The random matrix model.

The observation.

$M \times N$ matrix $\mathbf{Y}_N = \mathbf{A}_N \mathbf{S}_N + \mathbf{V}_N$, $M < N$.

- $\mathbf{S}_N$ non observable deterministic $K \times N$ matrix, $K < M$, $\text{rank}(\mathbf{S}_N) = K$. Rows of $\mathbf{S}_N$ represent K source signals.
- $\mathbf{A}_N$ deterministic $N \times K$ matrix $\mathbf{A}_N = \begin{bmatrix} \mathbf{a}_N(\varphi_1) & \cdots & \mathbf{a}_N(\varphi_K) \end{bmatrix}$ with

$$\mathbf{a}_N(\varphi) = \frac{1}{\sqrt{M}} \begin{bmatrix} 1 \\ e^{\imath\varphi} \\ \vdots \\ e^{\imath(M-1)\varphi} \end{bmatrix}$$

- $\mathbf{V}_N$ $M \times N$ complex Gaussian random matrix with i.i.d. entries, zero mean, variance σ^2.

Estimation of $(\varphi_k)_{k=1,\dots,K}$ from $\mathbf{Y}_N$.

The subspace method

Let $\mathbf{\Pi}_N$ be the orthogonal projection matrix on the range of $\mathbf{AA}^*$, or equivalently, on the K–dimensional eigenspace associated with the K greastest eigenvalues ($> \sigma^2$) of matrix

$$\mathbb{E}\left(\frac{\mathbf{YY}^*}{N}\right) = \mathbf{A}\frac{\mathbf{SS}^*}{N}\mathbf{A}^* + \sigma^2 \mathbf{I}_M$$

Let $\mathbf{\Pi}_N^\perp = \mathbf{I}_M - \mathbf{\Pi}_N$ be the orthogonal projector on the kernel of $\mathbf{AA}^*$, or equivalently on the eigenspace of $\mathbb{E}\left(\frac{\mathbf{YY}^*}{N}\right)$ associated to σ^2

Subspace estimation algorithm principle

$$\eta_N(\varphi) = \mathbf{a}_N(\varphi)^* \mathbf{\Pi}_N^\perp \mathbf{a}_N(\varphi) = 0 \quad \Leftrightarrow \quad \varphi \in \{\varphi_1, \ldots, \varphi_K\}.$$

$\varphi \to \eta_N(\varphi)$ is called the localization function.

Subspace estimation algorithm

Traditional subspace estimation

Angles are estimated as the argmin of the K smallest local minima $(\hat{\varphi}_{t,k})_{k=1,\ldots,K}$ of

$$\varphi \longrightarrow \hat{\eta}_{t,N}(\varphi) = \mathbf{a}_N(\varphi)^* \widehat{\mathbf{\Pi}}_N^{\perp} \mathbf{a}_N(\varphi)$$

where $\widehat{\mathbf{\Pi}}_N^{\perp}$ is the orthogonal projection matrix on the eigenspace associated with the $M-K$ smallest eigenvalues of $\frac{\mathbf{YY}^*}{N}$.

$\varphi \longrightarrow \hat{\eta}_{t,N}(\varphi)$ is the traditional estimated localization function.

Properties of the estimates when M is fixed and $N \to \infty$.

- $\| \frac{\mathbf{YY}^*}{N} - \left(\mathbf{A} \frac{\mathbf{SS}^*}{N} \mathbf{A}^* + \sigma^2 \mathbf{I}_M \right) \| \to 0$
- $\| \widehat{\mathbf{\Pi}}_N^{\perp} - \mathbf{\Pi}_N^{\perp} \| \to 0$
- $\sup_{\varphi} \left| \mathbf{a}_N(\varphi)^* \widehat{\mathbf{\Pi}}_N^{\perp} \mathbf{a}_N(\varphi) - \mathbf{a}_N(\varphi)^* \mathbf{\Pi}_N^{\perp} \mathbf{a}_N(\varphi) \right| \to 0$ almost surely
- $\hat{\varphi}_{t,k} - \varphi_k \to 0$
- $N^{1/2} \left[\frac{d}{d\varphi} \left(\mathbf{a}_N(\varphi)^* \widehat{\mathbf{\Pi}}_N^{\perp} \mathbf{a}_N(\varphi) \right) \right]_{\varphi = \varphi_k}$ as a zero mean Gaussian behaviour
- $N^{1/2} (\hat{\varphi}_{t,k} - \varphi_k)$ has a zero mean Gaussian behaviour

When M and N are of the same order of magnitude ?

We consider the asymptotic regime:

$M \to +\infty$, $N \to +\infty$, $\frac{M}{N} \to c$, $c > 0$. It is assumed that $c < 1$.

$\|\widehat{\mathbf{\Pi}}_N^{\perp} - \mathbf{\Pi}_N^{\perp}\|$ does not converge towards 0.

For each φ, $\hat{\eta}_{t,N}(\varphi) = \mathbf{a}_N(\varphi)^* \widehat{\mathbf{\Pi}}_N^{\perp} \mathbf{a}_N(\varphi)$ is not a consistent estimate of $\eta_N(\varphi) = \mathbf{a}_N(\varphi)^* \mathbf{\Pi}_N^{\perp} \mathbf{a}_N(\varphi)$.

- Derivation of uniformly consistent estimates of $\hat{\eta}_N(\varphi)$ of $\eta_N(\varphi)$:

$$\sup_{\varphi} |\hat{\eta}_N(\varphi) - \eta_N(\varphi)| \to 0$$

- Properties of "improved" estimates of the angles defined as the argmin of the K smallest local minima of $\varphi \to \hat{\eta}_N(\varphi)$: consistency and central limit theorems.

The various relevant regimes.

- K does not scale with M and N : In practice, $\frac{K}{M} << 1$.
- K scales with M and N : In practice $\frac{K}{M}$ is not "very small".
- The angles $(\varphi_k)_{k=1,\ldots,K}$ are fixed parameters that do not depend on M, N : vectors $(\mathbf{a}_N(\varphi_k))_{k=1,\ldots,K}$ are nearly orthogonal:

$$| < \mathbf{a}_N(\varphi_k), \mathbf{a}_N(\varphi_l) > | = \frac{1}{M} \left| \frac{\sin\left(M(\varphi_k - \varphi_l)\right)}{\varphi_k - \varphi_l} \right|$$

  Easy to estimate the $(\varphi_k)_{k=1,\ldots,K}$. Regime representing practical situations in which the angles are significantly different.
- For certain values of k, $\varphi_{k+1} - \varphi_k = \mathcal{O}(\frac{1}{M})$: more difficult estimation problem. Regime representating practical situations in which certain angles are close one from each others.

Overview of the results I.

If K does not scale with M and N: the spiked model case.

- If the noise variance is less than a certain threshold, the derivation of a uniformly consistent estimate $\hat{\eta}_{s,N}(\varphi)$ of $\eta_N(\varphi)$ is easy.
- If the angles are fixed: the traditional subspace angle estimates are consistent, and behave as the improved estimates $(\hat{\varphi}_{s,k})_{k=1,\ldots,K}$ (same speed of convergence $\frac{1}{N^{3/2}}$).
- If $\varphi_2 - \varphi_1 = \mathcal{O}(\frac{1}{M})$:
  - $(\hat{\varphi}_{t,i})_{i=1,2}$ not necessarily defined (unexistence of 2 local minima); if well defined, $N(\hat{\varphi}_{t,i} - \varphi_i)$ for $i = 1,2$ do not converge towards 0.
  - $N(\hat{\varphi}_{s,i} - \varphi_i)$ for $i = 1,2$ converge to 0
  - $\frac{1}{N^{1/2}} \left[\frac{d}{d\varphi} \left(\hat{\eta}_{s,N}(\varphi) \right) \right]_{\varphi=\varphi_i}$ for $i = 1,2$ has a zero mean Gaussian behaviour.
  - $N^{3/2}(\hat{\varphi}_{s,i} - \varphi_i)$ have a zero mean Gaussian behaviour for $i = 1,2$.

Overview of the results II.

If K may scale with M and N.

- If the noise variance is small enough, it is possible to derive a uniformly consistent estimate $\hat{\eta}_N(\varphi)$ of $\eta_N(\varphi)$.
- Estimating all the angles consistently by minimizing $\hat{\eta}_N(\varphi)$ when K scales with M, N seems difficult.
- $\frac{1}{N^{1/2}} \left[\frac{d}{d\varphi} \left(\hat{\eta}_N(\varphi) \right) \right]_{\varphi = \varphi_k}$ has a zero mean Gaussian behaviour.
- If $\hat{\varphi}_k$ is a local minimum of $\hat{\eta}_N(\varphi)$ and if $N(\hat{\varphi}_k - \varphi_k) \rightarrow 0$, then, $N^{3/2}(\hat{\varphi}_k - \varphi_k)$ has a zero mean Gaussian behaviour.
- In practice, for finite values of M, N, K for which $\frac{K}{M}$ is not very small:
  - The angles $(\hat{\varphi}_k)_{k=1,\ldots,K}$ minimizing $\hat{\eta}_N(\varphi)$ provide better results than the estimators $(\hat{\varphi}_{s,k})_{k=1,\ldots,K}$
  - The above mentioned Gaussian behaviour of $N^{3/2}(\hat{\varphi}_k - \varphi_k)$ is in practice a good prediction of the performance of the estimator $\hat{\varphi}_k$.

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Notations.

Spectral factorizations:

$$\mathbf{A}_N \frac{\mathbf{S}_N \mathbf{S}_N^*}{N} \mathbf{A}_N^* = \begin{bmatrix} \mathbf{u}_{1,N} & \cdots & \mathbf{u}_{K,N} \end{bmatrix} \begin{bmatrix} \lambda_{1,N} & & \\ & \ddots & \\ & & \lambda_{K,N} \end{bmatrix} \begin{bmatrix} \mathbf{u}_{1,N} & \cdots & \mathbf{u}_{K,N} \end{bmatrix}^*$$

where $\lambda_{1,N} \geq \cdots \geq \lambda_{K,N}$.

Assuming $N \geq M$

$$\frac{\mathbf{Y}_N \mathbf{Y}_N^*}{N} = \begin{bmatrix} \hat{\mathbf{u}}_{1,N} & \cdots & \hat{\mathbf{u}}_{M,N} \end{bmatrix} \begin{bmatrix} \hat{\lambda}_{1,N} & & \\ & \ddots & \\ & & \hat{\lambda}_{M,N} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{u}}_{1,N} & \cdots & \hat{\mathbf{u}}_{M,N} \end{bmatrix}^*$$

where $\hat{\lambda}_{1,N} \geq \cdots \geq \hat{\lambda}_{M,N}$.

Behaviour of the largest eigenvalues of $\frac{\mathbf{Y}_N \mathbf{Y}_N^*}{N}$

Assumption

For each $k = 1, \ldots, K$, $\lambda_{k,N} \to \rho_k$, and $\rho_1 > \rho_2 > \ldots > \rho_K > 0$.

Theorem (Benaych-Rao)

If $\rho_K > \sigma^2 \sqrt{c}$, then, for $k = 1, \ldots, K$,

$$\hat{\lambda}_{k,N} \to \gamma_k = \frac{(\sigma^2 c + \rho_k)(\sigma^2 + \rho_k)}{\rho_k} > \sigma^2(1 + \sqrt{c})^2$$

and

$$\hat{\lambda}_{K+1,N} \to \sigma^2(1 + \sqrt{c})^2$$

Behaviour of the largest eigenvectors of $\frac{\mathbf{Y}_N \mathbf{Y}_N^*}{N}$

Theorem (Benaych-Rao)

If $\rho_K > \sigma^2 \sqrt{c}$, then, for $k = 1, \ldots, K$, for each deterministic unit norm vectors $\mathbf{d}_N, \mathbf{e}_N$, it holds that

$$\mathbf{d}_N^* \hat{\mathbf{u}}_{k,N} \hat{\mathbf{u}}_{k,N}^* \mathbf{e}_N - h(\hat{\lambda}_{k,N}) \mathbf{d}_N^* \mathbf{u}_{k,N} \mathbf{u}_{k,N}^* \mathbf{e}_N \rightarrow 0$$

where h is a function depending on the Stieltjes transform of the limit eigenvalue distribution (Marcenko-Pastur) of $\frac{\mathbf{V}_N \mathbf{V}_N}{N}^*$, and which satisfies $h(\rho_k) > 0$ for each k.

Consistent estimator of $\eta_N(\varphi) = \mathbf{a}_N(\varphi)^* \mathbf{\Pi}_N^\perp \mathbf{a}_N(\varphi)$.

Assume that $\rho_K > \sigma^2\sqrt{c}$. Then

$$\sum_{k=1}^{K} \frac{|\mathbf{a}_N(\varphi)^* \hat{\mathbf{u}}_{k,N}|^2}{h(\hat{\lambda}_{k,N})} - \mathbf{a}_N(\varphi)^* \mathbf{\Pi}_N \mathbf{a}_N(\varphi) \xrightarrow[N\to\infty]{\text{a.s.}} 0$$

$\mathbf{\Pi}_N = \sum_{k=1}^{M} \hat{\mathbf{u}}_{k,N} \hat{\mathbf{u}}_{k,N}^* - \mathbf{\Pi}_N^\perp$

Modification of the traditional estimator of $\eta_N(\varphi)$ for K fixed

$$\hat{\eta}_{s,N}(\varphi) = \mathbf{a}_N(\varphi)^* \hat{\mathbf{\Pi}}_N^\perp \mathbf{a}_N(\varphi) + \sum_{k=1}^{K} \left(1 - \frac{1}{h(\hat{\lambda}_{k,N})}\right) \mathbf{a}_N(\varphi)^* \hat{\mathbf{u}}_{k,N} \hat{\mathbf{u}}_{k,N}^* \mathbf{a}_N(\varphi)$$

$\hat{\eta}_{s,N}(\varphi)$ is a (uniformly) consistent estimator of $\eta_N(\varphi)$.

Consistency of the improved angular estimates for fixed angles.

$(I_k)_{k=1,\ldots,K}$ disjoint intervals containing the $(\varphi_k)_{k=1,\ldots,K}$.

For $k = 1, \ldots, K$, $\hat{\varphi}^{(s)}_{k,N} = \mathrm{Argmin}_{\varphi \in I_k} |\hat{\eta}_{s,N}(\varphi)|$

It holds that $N(\hat{\varphi}^{(s)}_{k,N} - \varphi_k) \to 0$.

Proof: $|\hat{\eta}_{s,N}(\hat{\varphi}^{(s)}_{k,N})| \leq |\hat{\eta}_{s,N}(\varphi_k)| \to |\eta_N(\varphi_k)| = 0$ As $\sup_\phi |\hat{\eta}_{s,N}(\varphi) - \eta_N(\varphi)| \to 0$, it holds that $\eta_N(\hat{\varphi}^{(s)}_{k,N}) \to 0$. As $\mathbf{A}_N^* \mathbf{A}_N \to \mathbf{I}_K$, we get that $\|\mathbf{a}_N(\hat{\varphi}^{(s)}_{k,N})^* \mathbf{A}_N\|^2 \to 1$.

Consistency of the traditional angular estimates for fixed angles I.

Behaviour of $\hat{\eta}_{t,N}(\varphi)$.

$\mathbf{a}_N(\varphi)^* \left(\sum_{k=1}^K \hat{\mathbf{u}}_{k,N} \hat{\mathbf{u}}_{k,N}^* - \sum_{k=1}^K h(\rho_k) \mathbf{u}_{k,N} \mathbf{u}_{k,N}^* \right) \mathbf{a}_N(\varphi) \rightarrow 0$ uniformly

$\hat{\eta}_{t,N}(\varphi) - \overline{\eta}_N(\varphi) \rightarrow 0$ uniformly

$$
\begin{aligned}
\overline{\eta}_N(\varphi) &= \mathbf{a}_N(\varphi)^* \left(\mathbf{I} - \sum_{k=1}^K h(\rho_k) \mathbf{u}_{k,N} \mathbf{u}_{k,N}^* \right) \mathbf{a}_N(\varphi) \\
&= 1 - \mathbf{a}_N(\varphi)^* \left(\sum_{k=1}^K h(\rho_k) \mathbf{u}_{k,N} \mathbf{u}_{k,N}^* \right) \mathbf{a}_N(\varphi)
\end{aligned}
$$

Consistency of the traditional angular estimates for fixed angles II.

For $k = 1, \ldots, K$, $\hat{\varphi}^{(t)}_{k,N} = \mathrm{Argmin}_{\varphi \in I_k} |\hat{\eta}_{t,N}(\varphi)|$

It holds that $N(\hat{\varphi}^{(t)}_{k,N} - \varphi_k) \rightarrow 0$.

Unformal proof:

$$\overline{\eta}_N(\hat{\varphi}^{(t)}_{k,N}) \simeq \hat{\eta}_{t,N}(\hat{\varphi}^{(t)}_{k,N} \leq \hat{\eta}_{t,N}(\varphi_k) \simeq \overline{\eta}_N(\varphi_k) \simeq 1 - h(\rho_k)$$

If $N(\hat{\varphi}^{(t)}_{k,N} - \varphi_k)$ does not converge towards 0, there exist subsequences extracted from $\overline{\eta}_N(\hat{\varphi}^{(t)}_{k,N})$ which converge towards values strictly greater than $1 - h(\rho_k)$. Contradiction.

Asymptotic distribution of $N^{3/2}(\hat{\varphi}_{k,N}^{(s)} - \varphi_k)$.

Classical approach.

$$\left(\frac{d}{d\varphi}\hat{\eta}_{s,N}\right)_{\varphi=\hat{\varphi}_{k,N}^{(s)}} = \hat{\eta}_{s,N}^{'}(\hat{\varphi}_{k,N}^{(s)}) = 0$$

$$N^{3/2}\left(\hat{\varphi}_{k,N}^{(s)} - \varphi_k\right) \simeq -\frac{\frac{1}{\sqrt{N}}\hat{\eta}_{s,N}^{'}(\varphi_k)}{\frac{1}{N^2}\hat{\eta}_{s,N}^{''}(\varphi_k)}$$

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3. K may scale with M, N.
   - Background on the empirical eigenvalue distribution of $\frac{\mathbf{Y}\mathbf{Y}^*}{N}$.
   - Consistent estimation of $\eta_N(\varphi)$
   - Gaussian behaviour of $\frac{1}{\sqrt{N}}\left[\frac{d}{d\varphi}\left(\hat{\eta}_N(\varphi)\right)\right]_{\varphi=\varphi_k}$

4. Some numerical results..

More convenient notations.

Definitions

$$\mathbf{B}_N = \mathbf{A}_N \frac{\mathbf{S}_N}{\sqrt{N}}$$

It is assumed that $\sup_N \|\mathbf{B}_N\| < +\infty$.

$$\mathbf{W}_N = \frac{\mathbf{V}_N}{\sqrt{N}}$$

$$\mathbf{\Sigma}_N = \frac{\mathbf{Y}_N}{\sqrt{N}}$$

$$\mathbf{\Sigma}_N = \mathbf{B}_N + \mathbf{W}_N$$

3. K may scale with M, N.
   - Background on the empirical eigenvalue distribution of $\frac{\mathbf{YY}^*}{N}$.
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Characterization of the limit eigenvalue distribution μ_N

Dozier-Silverstein 2007: It exists a deterministic probability measure μ_N carried by $\mathbb{R}^+$ such that

- $\frac{1}{M}\sum_{k=1}^{M} \delta(\lambda - \hat{\lambda}_{k,N}) - \mu_N \rightarrow 0$ weakly almost surely

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How to characterize μ_N

- Stieltjes transform $m_N(z) = \int_{\mathbb{R}^+} \frac{\mu_N(d\lambda)}{\lambda - z}$ defined on $\mathbb{C} - \mathbb{R}^+$
- $m_N(z) := \frac{1}{M}\mathrm{Tr}\mathbf{T}_N(z)$ with
- $\mathbf{T}_N(z) = \left(\frac{\mathbf{B_N B}_N^*}{1+\sigma^2 c_N m_N(z)} - \left[z(1+\sigma^2 c_N m_N(z)) - \sigma^2(1-c_N) \right] \mathbf{I}_M \right)^{-1}$.

$w_N(z) = z(1+\sigma^2 c_N m_N(z))^2 - \sigma^2(1-c_N)(1+\sigma^2 c_N m_N(z))$ is an important function.

Convergence results.

$\mathbf{Q}_N(z) = (\mathbf{\Sigma}_N \mathbf{\Sigma}_N^* - z\mathbf{I}_M)^{-1}$

- $\frac{1}{M}\mathrm{Tr}\mathbf{Q}_N(z) = \hat{m}_N(z) = m_N(z) + \mathcal{O}_P(\frac{1}{N}) = \frac{1}{M}\mathrm{Tr}\mathbf{T}_N(z) + \mathcal{O}_P(\frac{1}{N})$.

Convergence results.

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- $\frac{1}{M}\mathrm{Tr}\mathbf{Q}_N(z) = \hat{m}_N(z) = m_N(z) + \mathcal{O}_P(\frac{1}{N}) = \frac{1}{M}\mathrm{Tr}\mathbf{T}_N(z) + \mathcal{O}_P(\frac{1}{N})$.
- Hachem et al.(2010), for $\|\mathbf{d}_N\| = 1$,

$$\mathbf{d}_N^* \mathbf{Q}_N(z) \mathbf{d}_N = \mathbf{d}_N^* \mathbf{T}_N(z) \mathbf{d}_N + \mathcal{O}_P(\frac{1}{\sqrt{N}})$$

Characterization of the support $\mathcal{S}_N$ of μ_N.

Reformulation of Dozier-Silverstein 2007 in Vallet-Loubaton-Mestre-2010

- $\mathcal{S}_N = [x_{1,-}^{(N)}, x_{1,+}^{(N)}] \cup \ldots [x_{Q,-}^{(N)}, x_{Q,+}^{(N)}]$.
- The number of intervals $Q \leq K+1$.
- The $(x_{k,-}^{(N)}, x_{k,+}^{(N)})_{k=1,\ldots,Q}$ are defined as the positive extrema of a certain rational function.
- $x_{1,-}^{(N)} > 0$ $(M/N < 1)$
- $\sup_N x_{Q,+}^{(N)} < t_2^+ < +\infty$

The support when K is finite.

$\rho_K > \sigma^2\sqrt{c}$

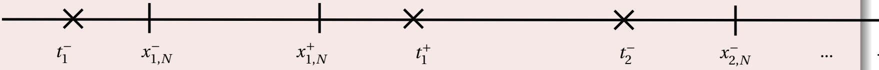
- $Q = K + 1$ intervals
- $[x_{1,N}^-, x_{1,N}^+] = [\sigma^2(1-\sqrt{c})^2 - o(1), \sigma^2(1+\sqrt{c})^2 + o(1)]$
- $[x_{k,N}^-, x_{k,N}^+] = [\gamma_k - o(1), \gamma_k + o(1)]$ for $k = 2, \ldots, K+1$

3. K may scale with M, N.
   - Background on the empirical eigenvalue distribution of $\frac{\mathbf{YY}^*}{N}$.
   - Consistent estimation of $\eta_N(\varphi)$
   - Gaussian behaviour of $\frac{1}{\sqrt{N}}\left[\frac{d}{d\varphi}\left(\hat{\eta}_N(\varphi)\right)\right]_{\varphi=\varphi_k}$

Separation of the "noise" and "signal" eigenvalues.

Generalization of the condition $\rho_K > \sigma^2 \sqrt{c}$ in the case K finite.

- $\limsup_N x_{1,N}^+ < \liminf_N x_{2,N}^-$



for all N large enough , t_1^-, t_1^+, t_2^- independent of N

- $\lambda_{K,N} > w_N(x_{2,N}^-)$ for all N large enough.

Consequences of the separation condition.

Consequences of the assumptions

- almost surely for N large enough

$$\hat{\lambda}_{K+1,N}, \ldots, \hat{\lambda}_{M,N} \in (t_1^-, t_1^+) \quad \text{and} \quad \hat{\lambda}_{1,N}, \ldots, \hat{\lambda}_{K,N} \in (t_2^-, t_2^+)$$

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- almost surely for N large enough

$$\hat{\lambda}_{K+1,N}, \ldots, \hat{\lambda}_{M,N} \in (t_1^-, t_1^+) \quad \text{and} \quad \hat{\lambda}_{1,N}, \ldots, \hat{\lambda}_{K,N} \in (t_2^-, t_2^+)$$

- almost surely for N large enough,

$$\hat{\omega}_{K+1,N}, \ldots, \hat{\omega}_{M,N} \in (t_1^-, t_1^+) \quad \text{and} \quad \hat{\omega}_{1,N}, \ldots, \hat{\omega}_{K,N} \in (t_2^-, t_2^+)$$

  with $\hat{\omega}_{1,N} \geq \ldots \geq \hat{\omega}_{M,N}$ the solutions of the equation $1 + \sigma^2 c_N \hat{m}_N(z) = 0$ with $\hat{m}_N(z) = \frac{1}{M} \mathrm{Tr} \mathbf{Q}_N(z)$

Contour integral representation of $\eta_N(\varphi)$.

The contour.

- For $y > 0$, we define the domain

$$\mathcal{R}_y = \left\{ u + iv : u \in [t_1^- - \delta, t_1^+ + \delta], v \in [-y, y] \right\}.$$

where $t_1^+ + \delta < t_2^-$.

Expression of $\mathbf{d}_N^* \mathbf{\Pi}_N^\perp \mathbf{e}_N$.

$$\mathbf{d}_N^* \mathbf{\Pi}_N^\perp \mathbf{e}_N = \frac{1}{2\pi \imath} \oint_{\partial \mathcal{R}_y^-} \mathbf{d}_N^* \mathbf{T}_N(z) \mathbf{e}_N \frac{w_N'(z)}{1 + \sigma^2 c_N m_N(z)} \mathrm{d}z$$

$\eta_N(\varphi)$ obtained for $\mathbf{d}_N = \mathbf{e}_N = \mathbf{a}_N(\varphi)$.

Consistent estimate of $\eta_N(\varphi)$.

$$\hat{\eta}_N(\varphi) = \frac{1}{2\pi\imath} \oint_{\partial \mathcal{R}_y^-} \mathbf{a}_N(\varphi)^* \mathbf{Q}_N(z) \mathbf{a}_N(\varphi) \frac{\hat{w}_N'(z)}{1+\sigma^2 c_N \hat{m}_N(z)} \mathrm{d}z$$

- Integral can be solved using the residue's theorem
- $\hat{\eta}_N = \mathbf{a}_N^* \left(\sum_{k=1}^M \hat{\xi}_{k,N} \hat{\mathbf{u}}_{k,N} \hat{\mathbf{u}}_{k,N}^* \right) \mathbf{a}_N$ with $(\hat{\xi}_{k,N})$ depending on $\hat{\lambda}_{1,N}, \ldots, \hat{\lambda}_{M,N}$ and $\hat{\omega}_{1,N}, \ldots, \hat{\omega}_{M,N}$.
- $\hat{\eta}_N(\varphi)$ depend on the $(\hat{\mathbf{u}}_{k,N} \hat{\mathbf{u}}_{k,N}^*)_{k=K+1,\ldots,M}$ and on the $(\hat{\mathbf{u}}_{k,N} \hat{\mathbf{u}}_{k,N}^*)_{k=1,\ldots,K}$

If K is finite.

$\frac{\hat{w}_N'(z)}{1+\sigma^2 c_N \hat{m}_N(z)} - \frac{w_{MP}'(z)}{1+\sigma^2 c m_{MP}(z)} \rightarrow 0$ uniformly on $\partial \mathcal{R}_y$ where $m_{MP}(z)$ is the Stieljes transform of the Marcenko-Pastur distribution associated to the noise part of the observation.

$$\hat{\eta}_{s,N}(\varphi) = \frac{1}{2\pi\imath} \oint_{\partial \mathcal{R}_y^-} \mathbf{a}_N(\varphi)^* \mathbf{Q}_N(z) \mathbf{a}_N(\varphi) \frac{w_{MP}'(z)}{1+\sigma^2 c m_{MP}(z)} \mathrm{d}z$$

3. K may scale with M, N.
   - Background on the empirical eigenvalue distribution of $\frac{\mathbf{YY}^*}{N}$.
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   - Gaussian behaviour of $\frac{1}{\sqrt{N}} \left[\frac{d}{d\varphi} \left(\hat{\eta}_N(\varphi) \right) \right]_{\varphi=\varphi_k}$

Gaussian behaviour of the consistent estimate of the quadratic form $\mathrm{Re}(\mathbf{d}_N^* \mathbf{\Pi}_N^{\perp} \mathbf{e}_N)$.

Notations.

$$\eta_N = \mathbf{d}_N^* \mathbf{\Pi}_N^{\perp} \mathbf{e}_N$$

$$\hat{\eta}_N = \frac{1}{2\pi\imath} \oint_{\partial \mathcal{R}_y^-} \mathbf{d}_N^* \mathbf{Q}_N(z) \mathbf{e}_N \frac{\hat{w}_N'(z)}{1 + \sigma^2 c_N \hat{m}_N(z)} \mathrm{d}z$$

$\psi_N(u) = \mathbb{E}\left(\exp iu\sqrt{N}\mathrm{Re}(\hat{\eta}_N - \eta_N)\right)$

Use of Gaussian tools (integration by parts formula, Poincaré-Nash inequality) to prove that $\psi_N(u)$ nearly satisfies a differential equation corresponding to the characteristic function of a zero Gaussian random variable.

As a function of $(\mathrm{Re}(\mathbf{W}_{i,j}), \mathrm{Im}(\mathbf{W}_{i,j}))_{(i=1,\ldots,M, j=1,\ldots,N}$, $\hat{\eta}_N$ does not meet the relevant regularity conditions.

Regularization of $\hat{\eta}_N$.

Introduce a differentiable regularization term.

- γ a smooth function equal to 1 on $[t_1^- - \epsilon, t_1^+ + \epsilon] \cup [t_2^- - \epsilon, t_2^+ + \epsilon]$ and 0 outside $[t_1^- - 2\epsilon, t_1^+ + 2\epsilon] \cup [t_2^- - 2\epsilon, t_2^+ + 2\epsilon]$.
- $\kappa_N = \Pi_{k=1}^M \gamma(\hat{\lambda}_{k,N})\gamma(\hat{\omega}_{k,N})$
- $\hat{\eta}_N = \hat{\eta}_N \kappa_N^2 + \mathcal{O}_P(\frac{1}{N^p})$ for each $p \in \mathbb{N}$
- As a function of $(\mathrm{Re}(\mathbf{W}_{i,j}), \mathrm{Im}(\mathbf{W}_{i,j}))_{(i=1,\ldots,M,j=1,\ldots,N}$, $\hat{\eta}_N \kappa_N^2$ meets the relevant technical conditions

$$\overline{\eta}_N = \frac{1}{2\pi \imath} \oint_{\partial \mathcal{R}_y^-} \mathbf{d}_N^* \mathbf{Q}_N(z) \mathbf{e}_N \frac{w_N'(z)}{1 + \sigma^2 c_N m_N(z)} \mathrm{d}z$$

$$\hat{\eta}_N = \overline{\eta}_N \kappa_N^2 + \mathcal{O}_P(\frac{1}{N})$$

Replace $\hat{\eta}_N$ by $\overline{\eta}_N \kappa_N^2$ into the Gaussian calculations.

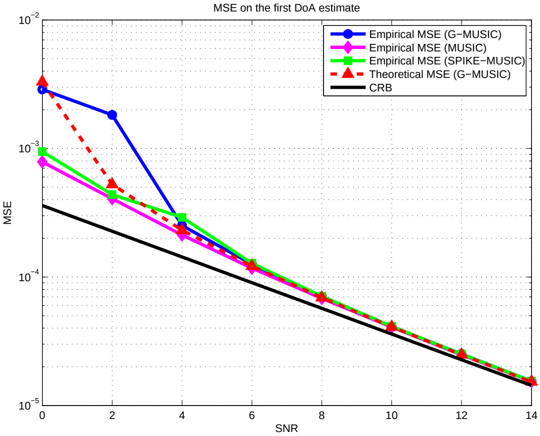
The last step.

Use the previous results when $\mathbf{d}_N = \frac{1}{N}\mathbf{a}_N(\varphi_k)', \mathbf{e}_N = \mathbf{a}_N(\varphi_k)$.

- $\eta_N = 0$ because $\mathbf{\Pi}_N^{\perp}\mathbf{a}_N(\varphi_k) = 0$
- $\sqrt{N}\left(\mathrm{Re}(\hat{\eta}_N - \eta_N)\right) = \frac{1}{\sqrt{N}}\left[\frac{d}{d\varphi}\left(\hat{\eta}_N(\varphi)\right)\right]_{\varphi=\varphi_k}$

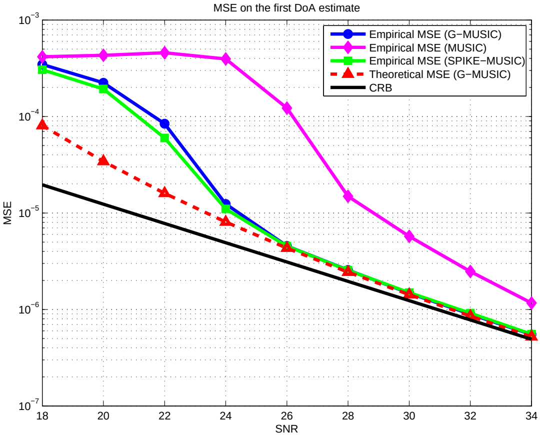
1. Problem statement and overview of the results.
2. The case K fixed.
3. K may scale with M, N.
4. Some numerical results..

$K = 2,\ M = 20, N = 40,\ \varphi_2 - \varphi_1 = \frac{\pi}{4}$.



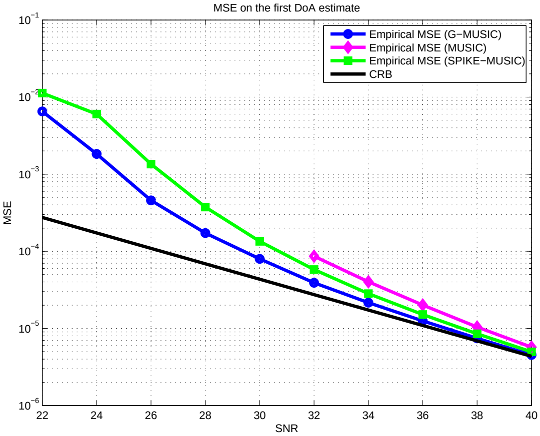
The minimum mean square error of the various estimates of φ_1 w.r.t. $10 \log_{10}(\frac{1}{\sigma^2})$.

$K = 2,\ M = 40, N = 80,\ \varphi_2 - \varphi_1 = \frac{\pi}{2M}$.



The minimum mean square error of the various estimates of φ_1 w.r.t. $10\log_{10}(\frac{1}{\sigma^2})$.

$K = 5,\ M = 20, N = 40,\ \varphi_{k+1} - \varphi_k = \frac{2\pi}{35}.$



The minimum mean square error of the various estimates of φ_1 w.r.t. $10\log_{10}(\frac{1}{\sigma^2})$.