The strong asymptotic freeness of large random and deterministic matrices

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Statement of results

No eigenvalues outside a neighborhood of the lim. support

Consider the N by N' so called "separable covariance matrix"

$$H_{N,N'} = A_N X_{N,N'} B_{N'} X^*_{N,N'} A_N, \text{ where}$$

- $\sqrt{N'} X_{N,N'}$: size $N \times N'$ with i.i.d. standard entries $\sim \mu$,
- $A_N, B_N \geq 0$: size $N \times N$ and $N' \times N'$ resp., s.t. $\mathcal{L}_{A_N} \to \mathcal{L}_a$, $\mathcal{L}_{B_{N'}} \to \mathcal{L}_b$.

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Theorem: Boutet de Mondvel, Khorunzhy and Vasilchuck (96)

As $N, N' \to \infty$ with $c_{N,N'} = \frac{N}{N'} \to c > 0$, $\mathcal{L}_{H_{N,N'}} \to \mu^{(c)}_{\mathcal{L}_a, \mathcal{L}_b}$ a.s.

Theorem: Bai and Silverstein (98), Paul and Silverstein (09)

If moreover μ has a finite fourth moment and for N large enough, $\mathrm{Supp}\ \mu^{(c_{N,N'})}_{\mathcal{L}_{A_N}, \mathcal{L}_{B_N}} \subset \mathrm{Supp}\ \mu^{(c)}_{\mathcal{L}_a, \mathcal{L}_b}$, then, a.s. $\forall \varepsilon$ and for N large enough,

$$\mathrm{Sp}\ H_{N,N'} \subset \mathrm{Supp}\ \mu^{(c)}_{\mathcal{L}_a, \mathcal{L}_b} + (-\varepsilon, \varepsilon).$$

Soft version

Theorem : M. (11), Collins, M. (11)

- X_N $N \times N$ GUE matrix,
- U_N $N \times N$ Haar matrix on $\mathcal{U}_N$,
- $\mathbf{Y}_N = (Y_1^{(N)}, \dots, Y_p^{(N)})$ arbitrary random $N \times N$ matrices,
- X_N, U_N and $\mathbf{Y}_N$ being independent.

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- X_N, U_N and $\mathbf{Y}_N$ being independent.

Assume that for any Hermitian matrix $H_N = P(\mathbf{Y}_N, \mathbf{Y}_N^*)$,

1. **Convergence of the empirical eigenvalues distribution**
   a.s. $\mathcal{L}_{H_N} \underset{N \to \infty}{\longrightarrow} \mathcal{L}_h$ with compact support,
2. **Convergence of the support**
   a.s. for N large enough, $\mathrm{Sp}\ H_N \subset \mathrm{Supp}\ \mathcal{L}_h + (-\varepsilon, \varepsilon)$

Then, almost surely, the same properties hold for any Hermitian matrix

$$H_N = P(X_N, U_N, U_N^*, \mathbf{Y}_N, \mathbf{Y}_N^*).$$

Non commutative probability space

Definition : $\mathcal{C}^*$-probability space $(\mathcal{A}, \cdot^*, \tau, \| \cdot \|)$

$\mathcal{A}$: $\mathcal{C}^*$-algebra,

$\cdot^*$: antilinear involution such that $(ab)^* = b^* a^* \ \forall a, b \in \mathcal{A}$,

τ : linear form such that

- $\tau[\mathbf{1}] = 1$,
- τ is tracial: $\tau[ab] = \tau[ba] \ \forall a, b \in \mathcal{A}$,
- τ is a faithful state: $\tau[a^* a] \geq 0, \forall a \in \mathcal{A}$ and vanishes iff $a = 0$.

Examples

- Commutative space: Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, consider $(L^\infty(\Omega, \mu), \bar{\cdot}, \mathbb{E}, \| \cdot \|_\infty)$,
- Matrix spaces: $(\mathrm{M}_N(\mathbb{C}), \cdot^*, \tau_N := \frac{1}{N}\mathrm{Tr}, \| \cdot \|)$.

Non commutative random variables

Proposition

If $aa^* = a^*a$ then there exists a compactly supported probability measure μ_a on $\mathbb{C}$ such that $\forall P$ polynomial $\tau\big[P(a, a^*)\ \big] = \int P(z, \bar{z}) d\mu_a(z)$. Moreover $\|a\| = \sup\{|t| \mid t \in \mathrm{Supp}\ \mu_a$. If A_N is an N by N normal matrix, then $\mu_{A_N} = \mathcal{L}_{A_N}$.

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Definition

- The map $\tau_{\mathbf{a}} : P \mapsto \tau\big[P(\mathbf{a},\mathbf{a}^*)\ \big]$: law of $\mathbf{a} = (a_1, \dots, a_p)$.
- **Convergence in n.c. law $\mathbf{a}_N \to \mathbf{a}$**:

$$\tau\big[P(\mathbf{a}_N,\mathbf{a}_N^*)\ \big] \underset{N\to\infty}{\longrightarrow} \tau\big[P(\mathbf{a},\mathbf{a}^*)\big],\ \forall P,$$

- **Strong convergence in n.c. law $\mathbf{a}_N \to \mathbf{a}$**: CV in n.c. law and

$$\big\|P(\mathbf{a}_N,\mathbf{a}_N^*)\ \big\| \underset{N\to\infty}{\longrightarrow} \big\|P(\mathbf{a},\mathbf{a}^*)\big\|,\ \forall P.$$

Interest of this notion for large matrices

Let $\mathbf{A}_N = (A_1^{(N)}, \ldots, A_p^{(N)})$ be a family of N by N matrices, and $\mathbf{a} = (a_1, \ldots, a_p)$ in $(\mathcal{A}, .^*, \tau)$.

Then $\mathbf{A}_N \underset{N\to\infty}{\overset{\mathcal{L}^{n.c.}}{\longrightarrow}} \mathbf{a}_N \Leftrightarrow \forall H_N = P(\mathbf{A}_N, \mathbf{A}_N^*)$ Hermitian

$$\mathcal{L}_{H_N} \underset{N\to\infty}{\longrightarrow} \mu_h, \text{ where } h = P(\mathbf{a}_N, \mathbf{a}_N^*).$$

Moreover $\mathbf{A}_N \underset{N\to\infty}{\overset{\mathcal{L}^{n.c.}}{\longrightarrow}} \mathbf{a}_N$ strongly $\Leftrightarrow \forall H_N = P(\mathbf{A}_N, \mathbf{A}_N^*)$ Hermitian

$$\begin{cases} \mathcal{L}_{H_N} \underset{N\to\infty}{\longrightarrow} \mu_h, \text{ where } h = P(\mathbf{a}_N, \mathbf{a}_N^*), \\ \forall \varepsilon > 0, \forall N \text{ large, } \mathrm{Sp}\ H_N \subset \mathrm{Supp}\ \mu_h + (-\varepsilon, \varepsilon). \end{cases}$$

The relation of freeness

Definition of freeness

The sub-algebras $\mathcal{A}_1, \dots, \mathcal{A}_p$ are free iff

$$\Big(a_j \in \mathcal{A}_{i_j},\ i_j \neq i_{j+1}, \text{ and } \tau(a_j) = 0, \forall j \geq 1 \Big) \Rightarrow \tau(a_1 a_2 \dots a_n) = 0\ \forall n \geq 1.$$

Theorem : Voiculescu

- X_N $N \times N$ GUE matrix,
- U_N $N \times N$ Haar matrix on $\mathcal{U}_N$,
- $\mathbf{Y}_N = (Y_1^{(N)}, \dots, Y_r^{(N)})$ arbitrary random $N \times N$ matrices, uniformly bounded,
- X_N, U_N and $\mathbf{Y}_N$ being independent.

If $\mathbf{Y}_N \underset{N\to\infty}{\overset{\mathcal{L}^{n.c.}}{\longrightarrow}} \mathbf{y}$, then $(X_N, U_N, \mathbf{Y}_N) \underset{N\to\infty}{\overset{\mathcal{L}^{n.c.}}{\longrightarrow}} (x, u, \mathbf{y})$, where x, u and $\mathbf{y}$ are free.

The asymptotic freeness of large random matrices

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The strong asymptotic freeness of large random matrices

Theorem : Haagerup and Thorbjørnsen, 05

Let $\mathbf{X}_N = (X_1^{(N)}, \dots, X_p^{(N)})$ be independent GUE matrices. Then $\mathbf{X}_N \underset{N\to\infty}{\overset{\mathcal{L}^{n.c.}}{\longrightarrow}} \mathbf{x}$ strongly, where $\mathbf{x} = (x_1, \dots, x_p)$ family of free semi-circular n.c.r.v.

Let $\mathbf{Y}_N = (Y_1^{(N)}, \dots, Y_p^{(N)})$ arbitrary random $N \times N$ matrices, such that $\mathbf{Y}_N \underset{N\to\infty}{\overset{\mathcal{L}^{n.c.}}{\longrightarrow}} \mathbf{y}$ strongly

Theorem : M., 11, Collins, M., 11

Let X_N be a GUE matrix, U_N be a Haar matrix on $\mathcal{U}_N$, such that X_N, U_N and $\mathbf{Y}_N$ are independent. Then $(X_N, U_N, \mathbf{Y}_N) \underset{N\to\infty}{\overset{\mathcal{L}^{n.c.}}{\longrightarrow}} (x, u, \mathbf{y})$ strongly, where x semi-circular n.c.r.v., u Haar unitary n.c.r.v. and $x, u, \mathbf{y}$ are free.

(Non direct) consequence

Proposition: the sum of two Hermitian random matrices, Collins, M. (11)

Let A_N, B_N be two $N \times N$ independent Hermitian random matrices.
Assume that:

1. the law of one of the matrices is invariant under unitary conjugacy,
2. a.s. $\mathcal{L}_{A_N} \underset{N\to\infty}{\longrightarrow} \mathcal{L}_a$ and $\mathcal{L}_{B_N} \underset{N\to\infty}{\longrightarrow} \mathcal{L}_b$ compactly supported
3. a.s. the spectra of the matrices converges to the support of the limiting distribution.

Then, a.s. the spectrum of $A_N + B_N$ converges to the support of $\mu \boxplus \nu$, where $\boxplus$ denotes the free additive convolution.

Remark: We do not assume that (A_N, B_N) converges strongly !

(Non direct) consequence

Consider the N by N' separable covariance matrix

$$H_{N,N'} = A_N X_{N,N'} B_{N'} X^*_{N,N'} A_N,$$

where

- the common distribution μ of the entries of $\sqrt{N'} X_{N,N'}$ is Gaussian,
- $N = \alpha n$, $N' = \beta n$ so that $c_{N,N'} = \frac{N}{N'} = \frac{\alpha}{\beta} = c$.
- A_N and B_N converges strongly in n.c. law.

Then, a.s. for n large enough, no eigenvalues of $H_{N,N'}$ are outside a small neighborhood of the support of the limiting distribution

Idea of the proof

From $(X_N, \mathbf{Y}_N)$ to $(U_N, \mathbf{Y}_N)$

Based on a coupling (X_N, U_N) between a GUE and a Haar matrix:

• Let Z_N be a Hermitian matrix. If $(Z_N, \mathbf{Y}_N) \underset{N\to\infty}{\overset{\mathcal{L}^{n.c.}}{\longrightarrow}} (z, \mathbf{y})$ strongly and $f_N : \mathbb{R} \to \mathbb{C}$ CV uniformly to f, then $(f_N(Z_N), \mathbf{Y}_N) \underset{N\to\infty}{\overset{\mathcal{L}^{n.c.}}{\longrightarrow}} (f(z), \mathbf{y})$ strongly.

• Let $X_N = V_N \Delta_N V_N^*$ GUE matrix, F_N the cumulative function of its eigenvalues. Then, $F_N \underset{N\to\infty}{\longrightarrow} F$ uniformly and

$$H_N := F_N(X_N) = V_N F_N(\Delta_N) V_N^* = V_N \mathrm{Diag}\,\big(\frac{1}{N}, \dots, \frac{N}{N}\big) V_N^*.$$

• Let G_N^{-1} be the inverse cumulative function of the eigenvalues of a Haar matrix, independent of $X_N, \mathbf{Y}_N$. Then $G_N^{-1} \underset{N\to\infty}{\longrightarrow} G^{-1}$ uniformly and

$$U_N := G_N^{-1}(H_N)$$

is a Haar matrix.

The main steps for the convergence of $(X_N, \mathbf{Y}_N)$

Haagerup and Thorbjørnsen's method:

1. A linearization trick,
2. Uniform control of matrix-valued Stieltjes transforms,
3. Concentration argument.

The main steps for the convergence of $(X_N, \mathbf{Y}_N)$

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In this proof, we use an idea of Bai and Silverstein

1. A linearization trick, unchanged,
2. Uniform control of matrix-valued Stieltjes transforms, based on an "asymptotic subordination property",
3. An intermediate inclusion of spectrum, by Shlyakhtenko,
4. Concentration argument, no significant changes.

An equivalent formulation

A linearization trick

The convergence of spectrum: a.s. for every self adjoint polynomial P, $\forall \varepsilon > 0$ and N large

$$\mathrm{Sp}\big(\ P(X_N, \mathbf{Y}_N, \mathbf{Y}_N^*)\ \big) \subset \mathrm{Sp}\big(\ P(\mathbf{x}, \mathbf{y}, \mathbf{y}^*)\ \big) + (-\varepsilon, \varepsilon).$$

is equivalent to the convergence: a.s. $\forall k \geq 1$, for every self adjoint degree one polynomial L with coefficient in $\mathrm{M}_k(\mathbb{C})$, $\forall \varepsilon > 0$ and N large

$$\mathrm{Sp}\big(\ L(X_N, \mathbf{Y}_N, \mathbf{Y}_N^*)\ \big) \subset \mathrm{Sp}\big(\ L(\mathbf{x}, \mathbf{y}, \mathbf{y}^*)\ \big) + (-\varepsilon, \varepsilon).$$

Sum of block matrices $H_N = a \otimes X_N + \sum_j (b_j \otimes Y_j^{(N)} + b_j^* \otimes Y_j^{(N)*})$!
Based on operator spaces techniques (Arveson's theorem and dilation of operators).

Matricial Stieltjes transforms and $\mathcal{R}$-transforms

Let $(\mathcal{A}, .^*, \tau, \| \cdot \|)$ be a $\mathcal{C}^*$-probability space. Consider z in $\mathrm{M}_k(\mathbb{C}) \otimes \mathcal{A}$.

Definitions

- The $\mathrm{M}_k(\mathbb{C})$-valued Stieltjes transform of z is

$$\begin{array}{rccc} G_z : & \mathrm{M}_k(\mathbb{C})^+ & \to & \mathrm{M}_k(\mathbb{C}) \\ & \Lambda & \mapsto & (\mathrm{id}_k \otimes \tau_N)\Big[(\Lambda \otimes \mathbf{1} - z)^{-1} \Big]. \end{array}$$

- The amalgamated $\mathcal{R}$-transform over $\mathrm{M}_k(\mathbb{C})$ of z is

$$\begin{array}{rccc} \mathcal{R}_z : & U & \to & \mathrm{M}_k(\mathbb{C}) \\ & \Lambda & \mapsto & G_z^{(-1)}(\Lambda) - \Lambda^{-1}. \end{array}$$

The subordination property

Let x selfadjoint and $\mathbf{y} = (y_1, \dots, y_q)$ be elements of $\mathcal{A}$ and let a and $\mathbf{b} = (b_1, \dots, b_q)$ be $k \times k$ matrices, a Hermitian. Define

$$s = a \otimes x, \quad t = \sum_{j=1}^{q} b_j \otimes y_j + b_j^* \otimes y_j^*.$$

Proposition

If x is free from $\mathbf{y}$, then one has

$$G_{s+t}(\Lambda) = G_t\Big(\Lambda - \mathcal{R}_s\big(\ G_{s+t}(\Lambda)\ \big)\ \Big).$$

From x a semicircular n.c.r.v.

$$\mathcal{R}_s : \Lambda \mapsto a\Lambda a.$$

Stability under analytic perturbations

Recall the subordination property:

$$G_{s+t}(\Lambda) = G_t\Big(\Lambda - \mathcal{R}_s\big(\ G_{s+t}(\Lambda)\ \big)\ \Big)$$

If G satisfies

$$G(\Lambda) = G_t\Big(\Lambda - \mathcal{R}_s\big(G(\Lambda)\ \big)\ \Big) + \Theta(\Lambda),$$

where Θ is an analytic perturbation, then we get

$$\|G(\Lambda) - G_{s+t}(\Lambda)\| \leqslant \big(1 + c\ \|(\mathrm{Im}\ \Lambda)^{-1}\|^2\ \big)\ \|\Theta(\Lambda)\|.$$

An asymptotic subordination property

Let X_N be a GUE matrix, let $\mathbf{Y}_N = (Y_1^{(N)}, \dots, Y_q^{(N)})$ be deterministic matrices and let a and $\mathbf{b} = (b_1, \dots, b_q)$ be $k \times k$ matrices, with a Hermitian. Define

$$S_N = a \otimes X_N, \quad T_N = \sum_{j=1}^{q} (b_j \otimes Y_j^{(N)} + b_j^* \otimes Y_j^{(N)*}).$$

Proposition

One has

$$G_{S_N+T_N}(\Lambda) = G_{T_N}\Big(\Lambda - \mathcal{R}_s\big(G_{S_N+T_N}(\Lambda) \big) \Big) + \Theta_N(\Lambda),$$

with Θ_N an analytic perturbation.

A first try

Hence, with $\mathbf{y}$ the limit in law of $\mathbf{Y}_N$

$$\begin{cases} G_{s+t}(\Lambda) & = & G_t\Big(\Lambda - \mathcal{R}_s\big(\ G_{s+t}(\Lambda)\ \big)\ \Big), \\ G_{S_N+T_N}(\Lambda) & = & G_{T_N}\Big(\Lambda - \mathcal{R}_s\big(\ G_{S_N+T_N}(\Lambda)\ \big)\ \Big) + \Theta_N(\Lambda). \end{cases}$$

$\Rightarrow$ we get an estimate of $\| G_{S_N+T_N}(\Lambda) - G_{s+t}(\Lambda)\|$ only if we can control $\| G_{T_N}(\Lambda) - G_t(\Lambda)\|$.
$\Rightarrow$ with the concentration machinery we get the Theorem, but with unsatisfactory assumptions on $\mathbf{Y}_N$ ...

Bai and Silverstein idea, in the flavor of free probability

Put x and $\mathbf{Y}_N$ in a same $\mathcal{C}^*$-probability space, free from each other. Same idea as discussing on the measure $\mu^{(c_{N,N'})}_{\mathcal{L}_{A_N},\mathcal{L}_{B_N}}$. Then

$$
\begin{aligned}
G_{s+T_N}(\Lambda) &= G_{T_N}\Big(\Lambda - \mathcal{R}_s\big(\ G_{s+T_N}(\Lambda)\ \big)\ \Big), \\
G_{S_N+T_N}(\Lambda) &= G_{T_N}\Big(\Lambda - \mathcal{R}_s\big(\ G_{S_N+T_N}(\Lambda)\ \big)\ \Big) + \Theta_N(\Lambda).
\end{aligned}
$$

$\Rightarrow$ we get an estimate of $\| G_{S_N+T_N}(\Lambda) - G_{s+T_N}(\Lambda)\|$ without any additionnal assumption on $\mathbf{Y}_N$.

An theorem about norm convergence

Theorem: by Shlyakhtenko, in an appendix of M. (11)

Let $\mathbf{Y}_N \underset{N\to\infty}{\overset{\mathcal{L}^{n.c.}}{\longrightarrow}} \mathbf{y}$ strongly, x a semicircular n.c.r.v. free from $(\mathbf{Y}_N, \mathbf{y})$. Then,

$$(x, \mathbf{Y}_N) \underset{N\to\infty}{\overset{\mathcal{L}^{n.c.}}{\longrightarrow}} (x, \mathbf{y}).$$

$\Rightarrow$ Together with this estimate of $\| G_{S_N+T_N}(\Lambda) - G_{s+T_N}(\Lambda)\|$, the concentration machinery applies.

Thank you !