Singular integrals on fractal subsets of Heisenberg groups

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- $H : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is bounded
- $H^* : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is bounded;
- $H^* f(x) = \sup_{\epsilon>0} | \int_{|x-y|>\epsilon} \frac{f(y)}{y-x} dy|$

Cauchy transform

- μ a finite Borel measure on $\mathbb{C}$
- $C_\mu^* f(z) = \sup_{\epsilon>0} | \int_{|z-w|>\epsilon} \frac{f(w)}{w-z} d\mu w|$

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- $\int C_\mu^*(f)^2 d\mu \leq C \int |f|^2 d\mu$?

Regular sets and measures

μ is (Ahlfors-David) m-regular if

$r^m/C \leq \mu(B(x,r)) \leq Cr^m$ for all $x \in \mathrm{spt}\mu, 0 < r < \mathrm{diam}(\mathrm{spt}\mu)$.

E is m-regular if

$r^m/C \leq \mathcal{H}^m(E \cap B(x,r)) \leq Cr^m$ for all $x \in E, 0 < r < \mathrm{diam}(E)$.

Cauchy transform

Theorem (Mattila, Melnikov, Verdera, 1996, David,...)

Suppose μ is a 1-regular finite Borel measure on $\mathbb{C}$. Then $C^*_\mu : L^2(\mu) \to L^2(\mu)$ bounded if and only sptμ is contained in a 1- regular curve.

Examples: $C^*_\mu : L^2(\mathcal{H}^1 \lfloor E) \to L^2(\mathcal{H}^1 \lfloor E)$ is unbounded for all self-similar 1-dimensional sets E with open set condition such that E is not equal to a line segment

Removable sets

A compact set E in a metric space is removable for a function class $\mathcal{F}$ if whenever $E \subset U$, U open, every function $f \in \mathcal{F}$ defined in $U \setminus E$ has an extension $\tilde{f}$ to U belonging to $\mathcal{F}$.

Removable sets

Theorem (Mattila, Melnikov, Verdera, 1996)

Suppose $E \subset \mathbb{C}$ is a compact 1-regular set. Then the following are equivalent:
E is removable for bounded analytic functions
E is removable for Lipschitz harmonic functions
$\mathcal{H}^1(E \cap \Gamma) = 0$ *for every rectifiable curve* Γ

Examples: all self-similar 1-dimensional sets with open set condition not equal to a line segment
Much more general results were later proven by David, Nazarov, Treil, Volberg, Tolsa, and others

Riesz transforms

- $0 < m < n$, μ m-regular
- $R^*_{m,\mu} f(x) = \sup_{\epsilon > 0} |\int_{|x-y|>\epsilon} \frac{y-x}{|y-x|^{m+1}} f(y) d\mu y|, x \in \mathbb{R}^n$
- David and Semmes, 1991: $R^*_{m,\mu} : L^2(\mu) \to L^2(\mu)$ is bounded for uniformly rectifiable m-regular measures μ
- Conjecture: converse holds
- Vihtilä, 1996: $R^*_{m,\mu} : L^2(\mu) \to L^2(\mu)$ is not bounded if m is not an integer
- Mattila and Paramonov, 1995: if $R^*_{m,\mu} : L^2(\mu) \to L^2(\mu)$ is bounded, then μ has m-flat tangent measures μ almost everywhere

Tangent measures

If μ is an m-regular measure, ν is a tangent measure of μ at x if there is a sequence $r_i \to 0$ and $c > 0$ such that

$$cr_i^{-m} T_{x,r_i \sharp}\mu \to \nu \text{ weakly as } i \to \infty.$$

$$T_{x,r}(y) = (y - x)/r$$

ν is m-flat if it is a constant multiple of the Lebesgue measure on an m-plane.

Removable sets

- A compact set $E \subset \mathbb{R}^n$ is removable for Lipschitz harmonic functions, LH-removable, if for all open sets U with $E \subset U$ every Lipschitz function $f : U \to \mathbb{R}$ which is harmonic in $U \setminus E$ is harmonic in U.
- Suppose $E \subset \mathbb{R}^n$ is $(n-1)$-regular. If E is not LH-removable, then almost everywhere it must have flat tangent measures.
- Examples: all non-planar self-similar $(n-1)$-dimensional sets with open set condition are LH-removable

Riesz transforms and removability

The connections between removability and singular integrals stem from fundamental solutions:
The fundamental solution for the Laplacian in $\mathbb{R}^n$, $n \geq 3$, is

$$\Gamma_n(x) = c|x|^{2-n}.$$

Lipschitz means bounded gradient, hence Lipschitz harmonic functions relate to the boundedness of the singular integrals with the Riesz kernel

$$\nabla\Gamma_n(x) = c|x|^{-n}x = cR_{n-1}(x).$$

Heisenberg group $\mathbb{H}$

Heisenberg group $\mathbb{H}$ is $\mathbb{R}^3$ equipped with a non-abelian group structure, with a left invariant metric and with natural dilations.

Heisenberg group $\mathbb{H}$

- $\mathbb{H} = \mathbb{C} \times \mathbb{R}$, $p = (w, s), q = (z, t) \in \mathbb{H}$
- $p \cdot q = (w + z, s + t + 2Im(w\bar{z}))$
- $||p|| = (|z|^4 + t^2)^{1/4}$
- $d(p, q) = ||p^{-1} \cdot q|| = (|w - z|^4 + |s - t - 2Im(w\bar{z})|^2)^{1/4}$
- $\delta_r(p) = (rz, r^2 t)$
- $d(\delta_r(p), \delta_r(q)) = rd(p, q)$
- $d(p \cdot q_1, p \cdot q_2) = d(q_1, q_2)$
- $\dim_H \mathbb{H} = 4$

Riesz-type transforms in $\mathbb{H}$

- $0 < m < 4$, μ m-regular in $\mathbb{H}$
- $R_m(p) = (\frac{z}{||p||^{m+1}}, \frac{t}{||p||^{m+2}}), p \in \mathbb{H}$,
- $R^*_{m,\mu}f(p) = \sup_{\epsilon>0} || \int_{d(p,q)>\epsilon} R_m(p^{-1}q)f(q)d\mu q||, p \in \mathbb{H}$
- Chousionis and Mattila, 2009: if $R^*_{m,\mu} : L^2(\mu) \to L^2(\mu)$ is bounded, then m must be an integer, 1,2 or 3, and μ has μ almost everywhere tangent measures ν such that

$$\nu = \mathcal{H}^m \lfloor H, H \text{ a homogeneous subgroup of } G,$$

$$\nu = \lim_{i\to\infty} c r_i^{-m} T_{p,r_i \sharp}\mu,$$

$$T_{p,r}(q) = \delta_{1/r}(p^{-1}q)$$

Removable sets

Nothing

What removable sets?

Horizontal differential operators

- $p = (z, t) = (x + iy, t) \in \mathbb{H}$
- $X = \frac{\partial}{\partial x} + 2y\frac{\partial}{\partial t}$, $Y = \frac{\partial}{\partial y} - 2x\frac{\partial}{\partial t}$
- $\nabla_H = (X, Y)$
- $\Delta_H = X^2 + Y^2$
- f is Δ_H-harmonic if $\Delta_H f = 0$.

Fundamental solution and the kernel K

- The fundamental solution for $\Delta_H f = 0$ is $\Gamma(p) = c||p||^{-2}$.
- $\nabla_H \Gamma(p) = c(\frac{x(x^2+y^2)+yt}{||p||^6}, \frac{y(x^2+y^2)-xt}{||p||^6}), p = (x+iy, t)$
- $K(p) = \nabla_H \Gamma(p)$
- $T^*_{K,\mu} f(p) = \sup_{\epsilon>0} || \int_{d(p,q)>\epsilon} K(p^{-1}q) f(q) d\mu q||, p \in \mathbb{H}$
- $T^*_{K,\mu}$ is unbounded on $L^2(\mu)$ for many self-similar measures μ

Removable sets

- A compact set $E \subset \mathbb{H}$ is removable for Lipschitz Δ_H-harmonic functions if for all open sets U with $E \subset U$ every Lipschitz function $f : U \to \mathbb{R}$ which is Δ_H-harmonic in $U \setminus E$ is harmonic in U.
- If $\mathcal{H}^3(E) = 0$, then E is removable.
- If $\dim E > 3$, then E is not removable.
- Many self-similar 3-dimensional sets are removable for Lipschitz Δ_H-harmonic functions.

Metric groups

- (G, d) is a complete separable metric group with the following properties:
- The left translations $\tau_q : G \to G$, $\tau_q(x) = q \cdot x, x \in G$, are isometries for all $q \in G$.
- There exist dilations $\delta_r : G \to G, r > 0$, which are continuous group homomorphisms for which,
- $\delta_1 =$ identity,
- $d(\delta_r(x), \delta_r(y)) = rd(x, y)$ for $x, y \in G, r > 0$,
- $\delta_{rs} = \delta_r \circ \delta_s$.

Self-similar sets

Let $\mathcal{S} = \{S_1, \dots, S_N\}, N \geq 2$, be an iterated function system of similarities of the form $S_i = \tau_{q_i} \circ \delta_{r_i}$ where $q_i \in G, r_i \in (0,1)$ and $i = 1, \dots, N$. The self-similar set C is the invariant set with respect to $\mathcal{S}$, that is, the unique non-empty compact set such that

$$C = \bigcup_{i=1}^{N} S_i(C).$$

Suppose that the sets $S_i(C)$ are pairwise disjoint for $i = 1, \dots, N$. Then

$$0 < \mathcal{H}^s(C) < \infty \quad \text{for} \quad \sum_{i=1}^{N} r_i^s = 1,$$

and the measure $\mu = \mathcal{H}^s \lfloor C$ is s-regular.

Self-similar sets

Theorem

Let $K : G \setminus \{e\} \to \mathbb{R}$ be an s-homogeneous kernel, $K(\delta_r(x)) = r^{-s}K(x)$ and

$$T^*_{K,\mu}f(x) = \sup_{\epsilon>0} \left| \int_{d(x,y)>\epsilon} K(x^{-1}y)f(y)d\mu y \right|, x \in G.$$

If there exists a fixed point x_w for some $S_w = S_{w_1} \circ \cdots \circ S_{w_k}, w = (w_1, \ldots, w_k)$, such that

$$\int_{C \setminus S_w(C)} K(x_w^{-1}y)d\mathcal{H}^s y \neq 0,$$

then the maximal operator $T^*_{K,\mathcal{H}^s\lfloor C}$ is unbounded in $L^2(\mathcal{H}^s\lfloor C)$, moreover $\|T^*_K(1)\|_{L^\infty(\mathcal{H}^s\lfloor C)} = \infty$.

Thank you

Thank you Julien, Stephane and others for Monastir, Porquerolles and ????!