Analysis of a fractal boundary: the example of the Knopp function

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- Theoritical results on the study of geometry of fractal boundaries with the help of wavelet transform obtained in previous works

$\rightarrow$ Lack of models to do numerical computation and test the algorithms written in the setting of wavelet transforms.

Choosing a model: the Knopp function

- Let

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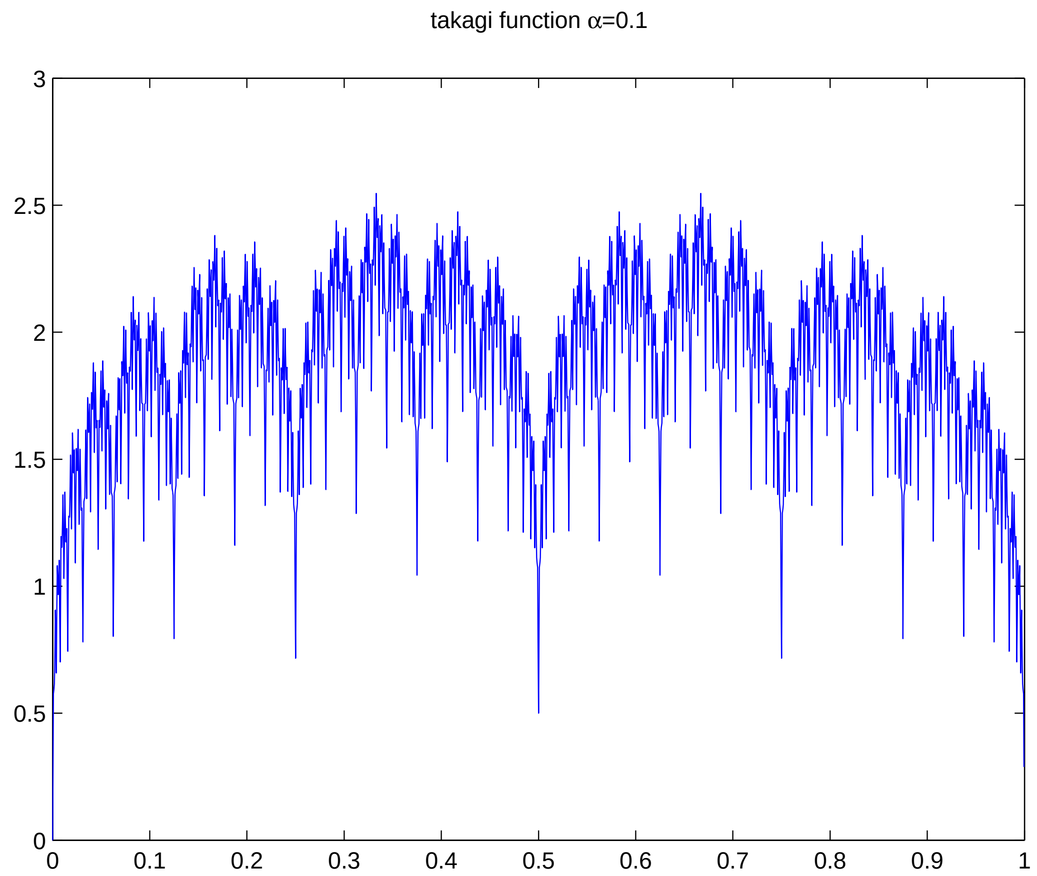
- Let $\alpha \in]0,1[$ and $F(x) = \sum_{j \geq 0} \sum_{k=0}^{2^j-1} 2^{-j\alpha} \Lambda(2^j x - k)$

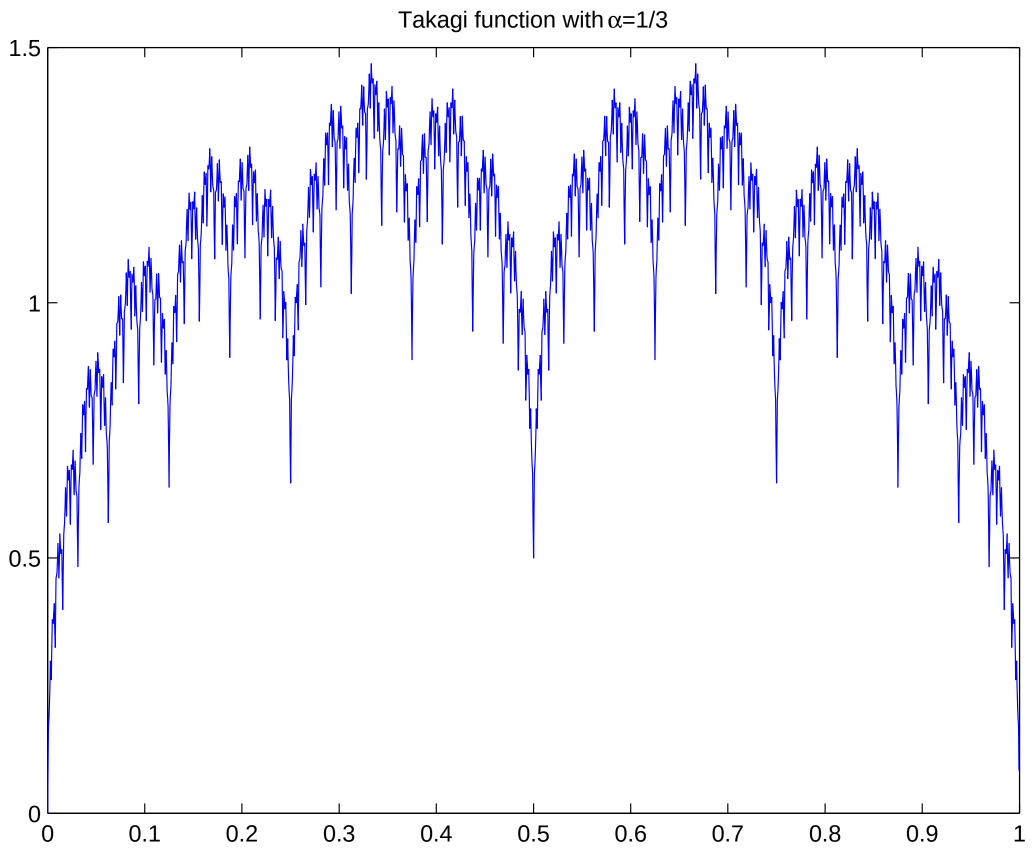
Several graphs

Let $\alpha = 0.1$.



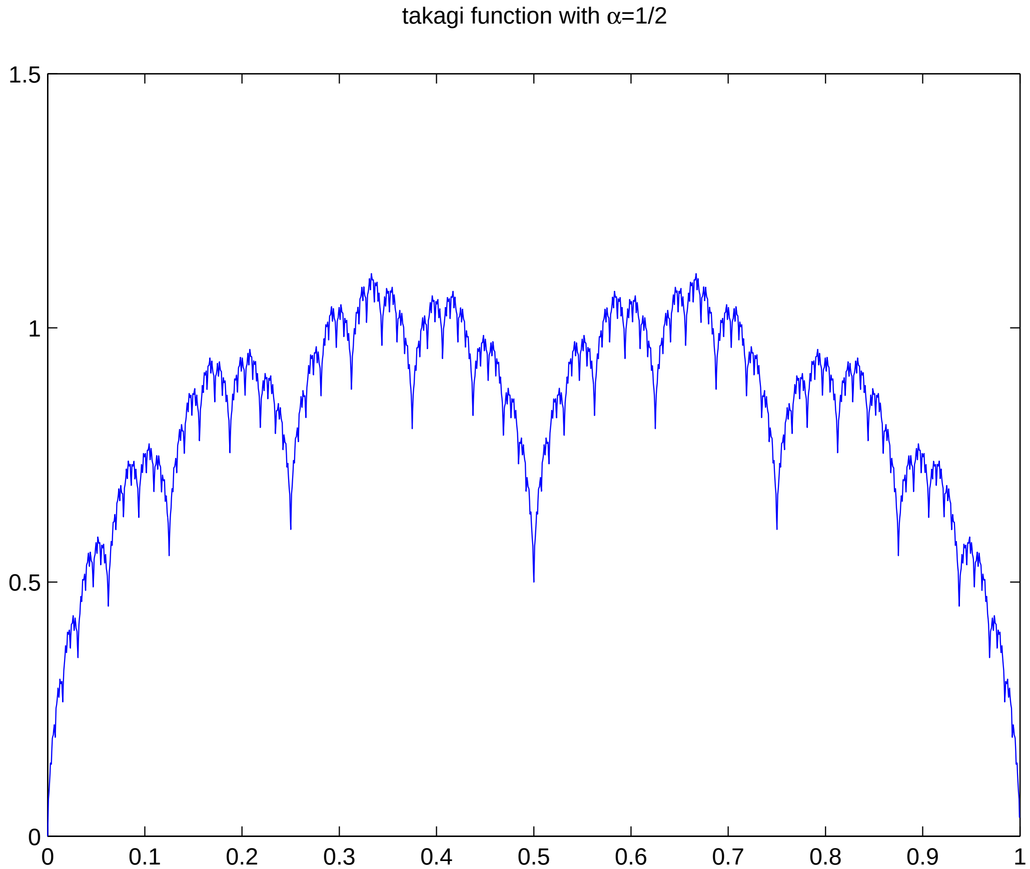
Several graphs

Let $\alpha = 1/3$.



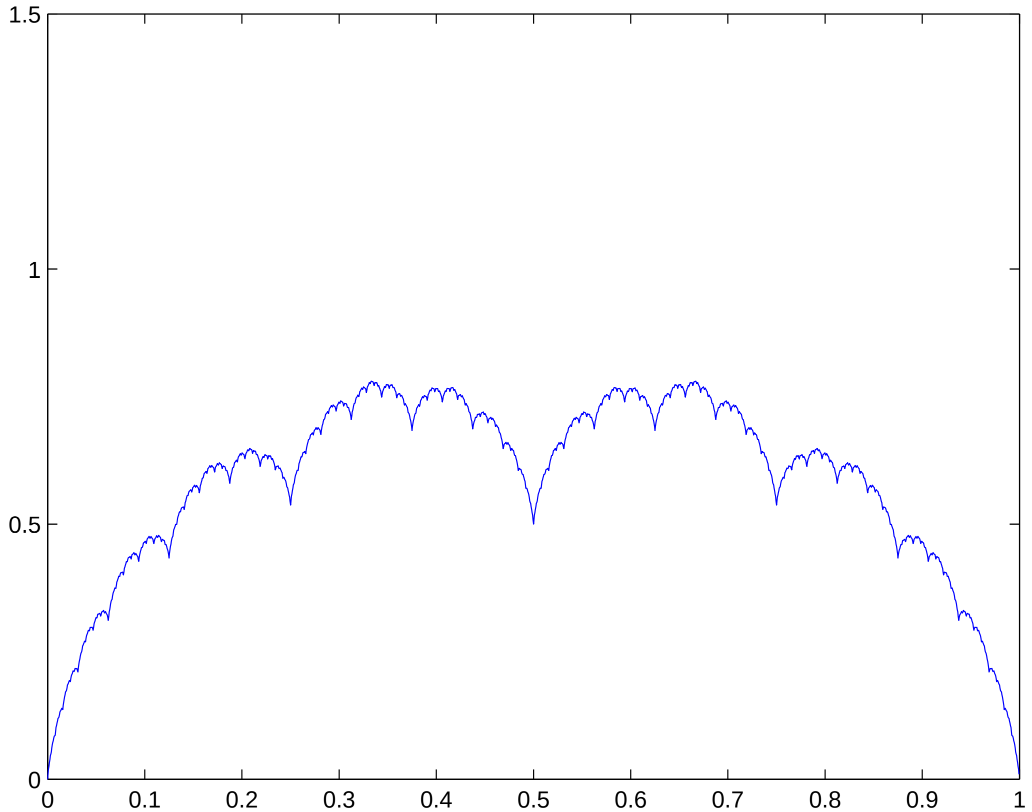
Several graphs

Let $\alpha = 0.5$.



Several graphs

Let $\alpha = 0.8$.



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Let start by making some remarks:

- Self similar function which satisfies

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- this class of functions is already well known from the point of view of pointwise regularity.
  - F is expanded on the Schauder basis. It yields $F \in C^{\alpha}([0,1])$.
  - Results of Ben Slimane and Jaffard (2001) in the more general setting of self similar functions:
    for all $x_0 \in [0,1]$, $h_F(x_0) = \alpha$.

p-exponent:

Definition:(Calderon and Zygmund 1961)
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$$\forall \rho \leq R : \quad \left(\frac{1}{\rho^d} \int_{|x - x_0| \leq \rho} |f(x) - P(x)|^p dx \right)^{\frac{1}{p}} \leq C \rho^u. \qquad (1)$$

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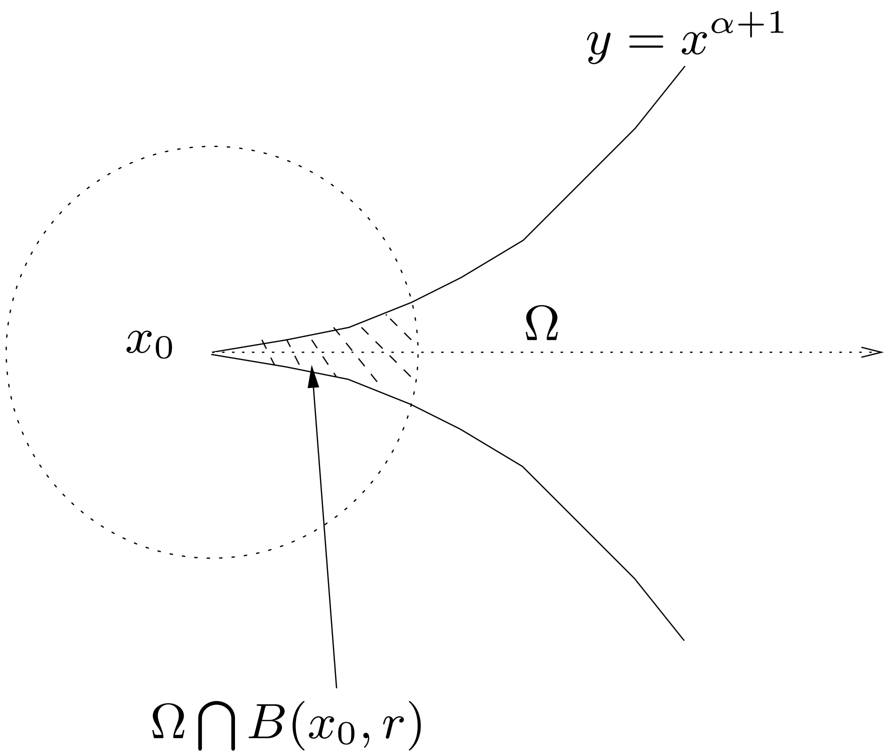
$\rightarrow$ the p-exponent of f at x_0 is $u^p_f(x_0) = \sup\{u : f \in T^p_u(x_0)\}$

Another regularity criterium:



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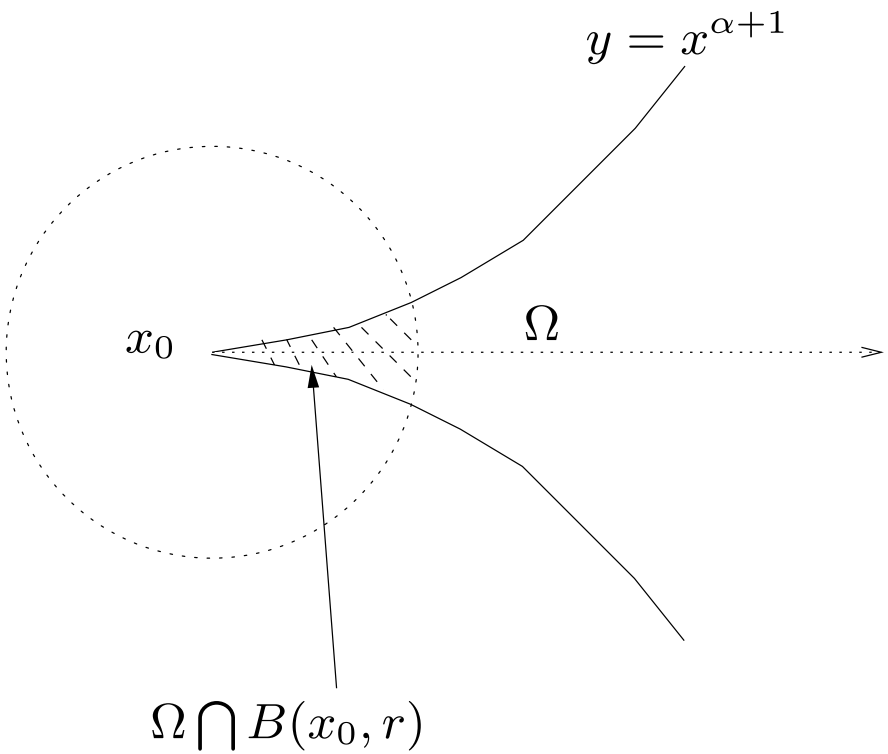

$$mes(\Omega \bigcap B(x_0, r)) = \int_{|x-x_0|\le\rho} |1_\Omega|^p dx \sim \int_{|t|\le\rho} |t|^{(\alpha+1)} dt \sim r^{\alpha+2}.$$

Adding a dimension

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Weak accessible points:

Definition: (Jaffard-Heurteaux 2006):
Let Ω a bounded open set of $\mathbb{R}^2$. Let a point X_0 at the boundary of Ω. The point X_0 is weak α-accessible in Ω if there exists $C > 0$ and $r_0 > 0$ such that $\forall r \leq r_0$,

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We call $E^w_{bil}(X_0)$ the supremum of the values such that there exists $C > 0$ and $r_0 > 0$ such that $\forall r \leq r_0$

$$\min \{Vol(\Omega^c \cap B(X_0, r)), Vol(\Omega \cap B(X_0, r))\} \leq C r^{\alpha+2} \ . \tag{3}$$

Strong accessible points:

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The infimum of all the values of α such that (4) holds is called the strong accessibility exponent in Ω at X_0. We denote it by $E^s_\Omega(X_0)$.

Proposition (Jaffard-Heurteaux 2006):

Let $X_0 \in \partial\Omega$.

$$E^w_\Omega(X_0) + 2 = \liminf_{r\to 0} \frac{\log\left(Vol(\Omega \cap B(X_0, r))\right)}{\log(r)}$$
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Let $f = 1_\Omega$

$$E^w_{bil}(X_0) = pu^p_f(X_0)$$

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- There is a set of points X_0 such that $E^s_{\Omega}(X_0) = E^s_{\Omega^c}(X_0) = \frac{1}{\alpha} - 1$.