Geometry of self-affine fractals

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Notation for sequences

Let $I_k = \{(i_1, \ldots, i_k) : 1 \leqslant i_j \leqslant m\}$ be the set of sequences of length k, where $k = 0, 1, 2, \ldots$.
Let $I = \cup_{k=0}^{\infty} I_k$ be the set of all finite sequences.
Let $I_\infty = \{(i_1 i_2 \ldots) : 1 \leqslant i_j \leqslant m\}$ be the corresponding set of infinite sequences.
We denote the number of terms in $\mathbf{i} \in I$ by $|\mathbf{i}|$.
We write $\mathbf{ij}$ for the sequence obtained by juxtaposition of the terms of $\mathbf{i}$ and $\mathbf{j}$.
Let $\mathbf{i} \wedge \mathbf{j} \in I$ denote the maximal common initial subsequence of both $\mathbf{i}$ and $\mathbf{j}$, for $\mathbf{i}, \mathbf{j} \in I_\infty$.
The *cylinders* $C_{\mathbf{i}} = \{\mathbf{j} \in I_\infty : \mathbf{i} \preceq \mathbf{j}\}$ for $\mathbf{i} \in I$ form a base of open and closed neighbourhoods for I_∞.

Almost self-affine sets

For each $\mathbf{i} = (i_1, \ldots, i_k) \in I_k$ let $\omega_{\mathbf{i}} = \omega_{i_1,\ldots,i_k} \in \mathbb{R}^N$ be a translation vector, and let $\omega = \{\omega_{\mathbf{i}} : \mathbf{i} \in I\}$ denote the family of such translations.
Let $T_1, \ldots, T_m$ be a set of linear contractions on $\mathbb{R}^N$. We assume throughout that there is some non-empty compact set $B \subset \mathbb{R}^N$ such that

$$T_{i_1}(B) + \omega_{i_1,\ldots,i_k} \subseteq B$$

for all $\mathbf{i} = (i_1, \ldots, i_k) \in I_k$ and for all ω under consideration. This ensures that each $\mathbf{i} = (i_1, i_2, \ldots) \in I_\infty$ determines a nested set of affine copies of B with intersection the single point

$$x^\omega(\mathbf{i}) \quad = \quad \omega_{i_1} + T_{i_1}\omega_{i_1,i_2} + T_{i_k}T_{i_2}\omega_{i_1,i_2,i_3} + \cdots . \tag{2.1}$$

We term the compact set E^ω given by the aggregate of these points,

$$E^\omega = \bigcup_{\mathbf{i} \in I_\infty} x^\omega(\mathbf{i}) \subset \mathbb{R}^N, \tag{2.2}$$

an *almost self-affine set*.

Self-affine sets

In the special case where

$$\omega_{i_1,\ldots,i_k} = \omega_{i_k} \qquad \text{for all } \mathbf{i} = (i_1, \ldots, i_k) \in I_k,$$

the set E^ω is the unique non-empty compact subset of $\mathbb{R}^N$ satisfying

$$E^\omega = \bigcup_{i=1}^{m} S_i(E^\omega) \tag{2.3}$$

where $S_i(x) = T_i(x) + \omega_i\,(i = 1, \ldots, m)$ are contracting affine transformations, so E^ω is the attractor of the iterated function system $\{S_1, \ldots, S_m\}$ and E^ω is a <u>self-affine</u> set.

Almost Self-affine measures and Self-affine measures

Given ‘probabilities’ $p_1, \ldots, p_m$ (so that $p_i > 0$ for each i and $\sum_{i=1}^m p_i = 1$), we define a Bernoulli measure μ on I_∞ by setting

$$\mu(C_{\mathbf{i}}) = p_{\mathbf{i}} \equiv p_{i_1} p_{i_2} \ldots p_{i_k} \quad (\mathbf{i} = i_1 \ldots i_k) \tag{2.4}$$

for each cylinder $C_{\mathbf{i}}$.

Let E^ω be an almost self affine set, the projected measure μ^ω on E^ω, defined by

$$\mu^\omega(A) = \mu\{\mathbf{i} : x^\omega(\mathbf{i}) \in A\} \qquad \text{for } A \subseteq \mathbb{R}^N \tag{2.5}$$

is termed an *almost self-affine measure*.

If E^ω is a self-affine set, the measure μ^ω is termed a *self-affine measure*, in which case

$$\mu^\omega(A) = \sum_{i=1}^m p_i \mu^\omega(S_i^{-1}(A)), \qquad \text{for } A \subseteq \mathbb{R}^N \tag{2.6}$$

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Dimensions of measures.

We denote the closed ball of radius r with center x by $B(x,r)$. Let ν be a Borel regular probability measure on $\mathbb{R}^n$. The *Hausdorff dimension* of ν is defined by

$$\dim_{\mathrm{H}} \nu = \sup\left\{ s : \liminf_{r\to 0} \frac{\log \nu(B(x,r))}{\log r} \geqslant s, \qquad \text{for } \nu\text{-almost all } x \right\},$$

The *packing dimension* of ν is defined by

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Generalized q-dimensions of measures.

Let $\mathcal{M}_r$ be the family of *r-mesh cubes* in $\mathbb{R}^N$, and ν be a Borel probability measure on $\mathbb{R}^N$.
For $q \neq 1$, the *lower* and *upper generalized q-dimensions* or *L^q-dimensions* of ν are defined by

$$\underline{D}^q(\nu) = \liminf_{r \to 0} \frac{\log \sum_{\mathcal{M}_r} \nu(C)^q}{(q-1)\log r}, \quad \overline{D}^q(\nu) = \limsup_{r \to 0} \frac{\log \sum_{\mathcal{M}_r} \nu(C)^q}{(q-1)\log r}.$$

For $q = 1$, $\underline{D}^1(\nu)$ and $\overline{D}^1(\nu)$ are defined by

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If $\underline{D}^q(\nu) = \overline{D}^q(\nu)$, the common value denoted by $D^q(\nu)$ is called the *generalized q-dimension*.

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Integral forms of generalized dimensions

For $q > 0, q \neq 1$

$$\underline{D}^q(\nu) = \liminf_{r\to 0} \frac{\log \int \nu(B(x,r))^{q-1} d\nu(x)}{(q-1)\log r},$$
$$\overline{D}^q(\nu) = \limsup_{r\to 0} \frac{\log \int \nu(B(x,r))^{q-1} d\nu(x)}{(q-1)\log r},$$

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Moreover, $\underline{D}^q(\nu)$ and $\overline{D}^q(\nu)$ are monotonic decreasing in q.

Proposition:

Let ν be a Borel probability measure on $\mathbb{R}^N$. Then for ν almost-all x

$$\lim_{q \searrow 1} \underline{D}^q(\nu) \leqslant \liminf_{r \to 0} \frac{\log \nu(B(x,r))}{\log r} \leqslant \limsup_{r \to 0} \frac{\log \nu(B(x,r))}{\log r} \leqslant \lim_{q \nearrow 1} \overline{D}^q(\nu). \quad (2.7)$$

We use it to show that

$$\limsup_{r \to 0} \frac{\log \mu^{\omega}(B(x,r))}{\log r}$$

has a sharp upper bound $(\min\{d_1, N\})$

Singular value function

Definition:

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a non-singular linear contraction. The singular values $\alpha_i \equiv \alpha_i(T)$ of T ($i = 1, \ldots, n$) are the positive square roots of the eigenvalues of T^*T, where T^* is the transpose or adjoint of T. Equivalently, they are the lengths of the principal semi-axes of the image $T(B)$ of the unit ball B. We adopt the convention that $1 > \alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_n > 0$.

Definition:

The singular value function $\phi^s(T)$ is defined for $0 \leqslant s \leqslant n$ as

$$\phi^s(T) = \alpha_1 \alpha_2 \cdots \alpha_{m-1} \alpha_m^{s-m+1},$$

where m is the integer such that $m - 1 < s \leqslant m$, with the convention that $\phi^s(T) = (\alpha_1 \alpha_2 \cdots \alpha_n)^{s/n}$ if $s \geqslant n$.

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Pressure function

For $s \geqslant 0$ and $q \geqslant 0, q \neq 1$ define:

$$\begin{aligned} P(s,q) &= \lim_{k\to\infty} \frac{1}{k} \frac{\log \sum_{|\mathbf{i}|=k} \phi^s(T_\mathbf{i})^{1-q}\mu(C_\mathbf{i})^q}{q-1} \\ &\equiv \lim_{k\to\infty} \frac{1}{k} \frac{\log \int \left(\phi^s(T_{\mathbf{i}|k})^{-1}\mu(C_{\mathbf{i}|k})\right)^{q-1} d\mu(\mathbf{i})}{q-1}, \end{aligned}$$

For $s \geqslant 0$ and $q = 1$,

$$\begin{aligned} P(s,1) &= \lim_{k\to\infty} \frac{1}{k} \sum_{|\mathbf{i}|=k} \mu(C_\mathbf{i}) \log\left(\phi^s(T_\mathbf{i})^{-1}\mu(C_\mathbf{i})\right) \\ &\equiv \lim_{k\to\infty} \frac{1}{k} \int \log\left(\phi^s(T_\mathbf{i})^{-1}\mu(C_{\mathbf{i}|k})\right) d\mu(\mathbf{i}). \end{aligned}$$

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Critical value

For each $q \geqslant 0$, there is a unique number $d_q > 0$ such that $P(d_q, q) = 0$; specifically the d_q satisfy

$$\begin{aligned} P(d_q, q) &= \lim_{k\to\infty} \frac{1}{k} \frac{\log \sum_{|\mathbf{i}|=k} \phi^{d_q}(T_\mathbf{i})^{1-q} \mu(C_\mathbf{i})^q}{q-1} = 0 \qquad (q \neq 1) \\ P(d_1, 1) &= \lim_{k\to\infty} \frac{1}{k} \log \sum_{|\mathbf{i}|=k} \mu(C_\mathbf{i}) \log \left(\phi^{d_1}(T_\mathbf{i})^{-1} \mu(\mathbf{i})\right) = 0. \end{aligned} \tag{3.8}$$

For $q \geqslant 0$, d_q is strictly monotonic decreasing in q.

d_q is continuous at all $q \neq 1$ and is upper semicontinuous at $q = 1$.

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Information dimension

Theorem:

Let μ be a Bernoulli measure on I_∞ given by (2.4) and let μ^ω be the projection of μ onto E^ω. Suppose that there are numbers $s < \min\{d_1, N\}$ that are arbitrarily close to $\min\{d_1, N\}$ for which there exists $c < \infty$ such that

$$\mathrm{E}\left(|x^\omega(\mathbf{i}) - x^\omega(\mathbf{j})|^{-s}\right) \leqslant \frac{c}{\phi^s(T_{\mathbf{i}\wedge\mathbf{j}})} \quad (\mathbf{i} \neq \mathbf{j} \in I_\infty).$$

Then, for almost all ω, the measure μ^ω is exact dimensional, with

$$\lim_{r\to 0} \frac{\log \mu^\omega(B(x,r))}{\log r} = D^1(\mu^\omega) = \min\{d_1, N\}$$

for μ^ω-almost all x.

Dimensions of self-affine measures.

Theorem:

For $S_i(x) = T_i(x) + \omega_i$ $(i = 1, 2, \ldots, m)$ let E^ω be the self-affine subset of $\mathbb{R}^N$ satisfying (2.3). Let μ be the Bernoulli measure on I_∞ given by (2.4) and let μ^ω be the projection of μ onto E^ω, that is the self-affine measure satisfying (2.6). Assume that $\|T_i\| < \frac{1}{2}$ for all $1 \leqslant i \leqslant m$. Then, for Nm-Lebesgue almost all $(\omega_1, \ldots, \omega_m) \in \mathbb{R}^{Nm}$, the measure μ^ω is exact dimensional, with

$$\lim_{r \to 0} \frac{\log \mu^\omega(B(x, r))}{\log r} = D^1(\mu^\omega) = \min\{d_1, N\}$$

for μ^ω-almost all x.

Dimensions of almost self-affine measures.

Theorem:

Let E^{ω} be the almost self-affine subset of $\mathbb{R}^N$. Let μ be the Bernoulli measure on I_∞ given by (2.4) and let μ^{ω} be the almost self-affine measure on E^{ω} given by (2.5). Suppose that $\|T_i\| < 1$ for all $1 \leqslant i \leqslant m$. Let D be a bounded region in $\mathbb{R}^N$ and let ¶ be a probability measure on Ω such that $\{\omega_{\mathbf{i}} : \mathbf{i} \in \Omega\}$ are independent identically distributed random vectors in D with a distribution that is absolutely continuous with respect to N-dimensional Lebesgue measure. Then, for ¶-almost all $\omega \in \Omega$, the measure μ^{ω} is exact dimensional with

$$\lim_{r \to 0} \frac{\log \mu^{\omega}(B(x,r))}{\log r} = D^1(\mu^{\omega}) = \min\{d_1, N\}$$

for μ^{ω}-almost all x.

Lower bound.

We use

$$\begin{aligned}
\mu^{\omega}(B(x^{\omega}(\mathbf{i}),r)) &= \mu(\mathbf{j} :| x^{\omega}(\mathbf{i}) - x^{\omega}(\mathbf{j}) | \leqslant r) \\
&= \mu\left(\mathbf{j} : r^{s}|x^{\omega}(\mathbf{i}) - x^{\omega}(\mathbf{j})|^{-s} \geqslant 1\right) \\
&\leqslant \int_{I_\infty} r^{s}|x^{\omega}(\mathbf{i}) - x^{\omega}(\mathbf{j})|^{-s} d\mu(\mathbf{j}).
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{E}\left(\mu^{\omega}(B(x^{\omega}(\mathbf{i}),r))\right) &\leqslant \mathrm{E}\int_{I_\infty} r^{s}|x^{\omega}(\mathbf{i}) - x^{\omega}(\mathbf{j})|^{-s} d\mu(\mathbf{j}) \\
&\leqslant r^{s}\int_{I_\infty} \frac{c}{\phi^{s}(T_{\mathbf{i}\wedge\mathbf{j}})} d\mu(\mathbf{j}) \\
&\leqslant c r^{s}\sum_{k=1}^{\infty}\phi^{s}(T_{\mathbf{i}|k})^{-1}\mu(C_{\mathbf{i}|k}).
\end{aligned}$$