Spectral shock waves in dynamical random matrix models

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Outline

- Trivia on real Burgers equation
- Large N limit of RMT and complex, inviscid Burgers
  1. Where are the shocks?
  2. What plays the role of spectral viscosity?
- Finite N as the inverse of viscosity in the spectral flow - Airy, Pearcey, Bessel and Bessoid functions as heralds of the shocks
- Optical analogies and applications
- Summary

Real Burgers equation

- $\partial_t f(x,t) + f(x,t)\partial_x f(x,t) = \mu \partial_{xx} f(x,t)$
  $f(x,t)$ is the velocity field at time t and position x of the fluid with viscosity μ.
- One-dimensional toy model for turbulence [Burgers 1939]
- But, equation turned out to be exactly integrable [Hopf 1950],[Cole 1951]
  If $f(x,t) = -2\mu\partial_x \ln d(x,t)$, then
  $\partial_t d(x,t) = \mu\partial_{xx} d(x,t)$ (diffusion equation), so general solution comes from Cole-Hopf transformation where

$$d(x,t) = \frac{1}{\sqrt{4\pi\mu t}} \int_{-\infty}^{+\infty} e^{-\frac{(x-x')^2}{4\mu t} - \frac{1}{2\mu}\int_0^{x'} d(x'',0)dx''} dx'$$

Inviscid real Burgers equation

- $\partial_t f(x,t) + f(x,t)\partial_x f(x,t) = 0$
  where $f(x,0) = f_0(x)$.
- Solution by the method of characteristics:
- If $x(t)$ is the solution of ODE $dx(t)/dt = f(x(t),t)$, then $F(t) \equiv f(x(t),t)$ is constant in time along characteristic curve on the (x,t) plane
- Then $dx/dt = F$ and $dF/dt = 0$ lead to $x(t) = x(0) + tF(0)$ and $F(t) = F(0)$
- Defining $\xi \equiv x(0)$ we get
  $f(x,t) = f(\xi,0) = f_0(\xi) = f_0(x - tf(x,t))$, i.e. implicit relation determining the solution of the Burgers equation.
- When $d\xi/dx = \infty$, we get the shock wave.

Inviscid real Burgers equation

- In the case of inviscid Burgers equation, characteristics are straight lines, but with different slopes (velocity depends on the position)
- Characteristics method fails when lines cross (shock wave)
- Finite viscosity (or diffusive constant) smoothens the shock
- Inviscid limit of viscid Burgers equation is highly non-trivial

Inviscid complex Burgers equation

After considerable and fruitless efforts to develop a Newtonian theory of ensembles, we discovered that the correct procedure is quite different and much simpler...... from F.J. Dyson, J. Math. Phys. 3 (1962) 1192

- $H_{ij} \to H_{ij} + \delta H_{ij}$ with $< \delta H_{ij} = 0 >$ and $< (\delta H_{ij})^2 >= (1 + \delta_{ij})\delta t$
- For eigenvalues x_i, random walk undergoes in the "electric field" (Dyson) $< \delta x_i >\equiv E(x_i)\delta t = \sum_{i \neq j} \left(\frac{1}{x_j - x_i} \right) \delta t$ and $< (\delta x_i)^2 >= \delta t$
- Resulting SFP equation for the resolvent in the limit $N = \infty$ and $\tau = Nt$ reads $\partial_\tau G(z,\tau) + G(z,\tau)\partial_z G(z,\tau) = 0$ where $G(z,\tau) = \frac{1}{N} \left\langle \mathrm{tr} \frac{1}{z - H(\tau)} \right\rangle$ is the resolvent
- Non-linear, inviscid complex Burgers (Hopf, Voiculescu) equation

Inviscid complex Burgers equation - details

- SFP eq:
  $\partial_t P(\{x_j\},t) = \frac{1}{2}\sum_i \partial_{ii}^2 P(\{x_j\},t) - \sum_i \partial_i (E(x_i)P(\{x_j\},t))$
- Integrating, normalizing densities to 1 and rescaling the time $\tau = Nt$ we get
  $\partial_\tau \rho(x) + \partial_x \rho(x) P.V. \int dy \frac{\rho(y)}{x-y} =$
  $\frac{1}{2N}\partial_{xx}^2 \rho(x) + P.V. \int dy \frac{\rho_c(x,y)}{x-y}$
- r.h.s. tends to zero in the large N limit
- $\frac{1}{x \pm i\epsilon} = P.V. \frac{1}{x} \mp i\pi\delta(x)$
- Taking Hilbert transform of the above equation and using above Sochocki formula converts pair of singular integral-differential equations onto complex inviscid Burgers equation.

Dolphins wisdom - surfing the shock wave

Tracing the singularities of the flow allows to understand the pattern of the evolution of the complex system without explicit solutions of the complicated hydrodynamic equations...

UK Daily Mail, July 11th 2007

Complex inviscid Burgers Equation

- Complex Burgers equation $\partial_\tau G + G \partial_z G = 0$
- Complex characteristics, trivial initial conditions
  $G(z,\tau) = G_0(\xi[z,\tau)])$ $\qquad G_0(z) = G(\tau = 0, z) = \frac{1}{z}$
  $\xi = z - G_0(\xi)\tau \quad (\xi = x - vt)$, so solution reads
  $G(z,\tau) = G_0(z - \tau G(z,\tau))$
- Shock wave when $\frac{d\xi}{dz} = \infty$
- Equivalently, $dz/d\xi = 0$, then $\xi_c = \pm\sqrt{\tau}$, so
  $z_c = \xi_c + G_0(\xi_c)\tau = \pm 2\sqrt{\tau}$
- Since explicit solution easily reads
  $G(z,\tau) = \frac{1}{2\pi\tau}(z - \sqrt{z^2 - 4\tau})$, i.e. $\rho(x,\tau) = \frac{1}{2\pi\tau}\sqrt{4\tau - x^2}$,
  we see that shock waves appear at the edges of the spectrum
  $(x = \pm 2\sqrt{\tau})$.

Where is the viscosity?

- Let us define $D_N(z,\tau) \equiv \langle \det(z - H(\tau)) \rangle$
- Opening the determinant with the help of auxilliary Grassmann variables and perfoming the averaging one gets easily
  $D_N(z,\tau) = \int \exp\left(\sum_i \bar{\eta}_i z \eta_i - \frac{\tau}{N} \sum_{i<j} \bar{\eta}_i \eta_i \bar{\eta}_j \eta_j\right) \prod_{l,r} d\bar{\eta}_l d\eta_r$
- Differentiating and using the properties of the Grassmann variables one gets that D_N obeys complex equation
  $\partial_\tau D_N(z,\tau) = -\frac{1}{2N} \partial_{zz} D(z,\tau)$.

Where is the viscosity? - cont.

- $\partial_\tau D_N(z,\tau) = -\frac{1}{2N}\partial_{zz} D(z,\tau)$.
- Then complex Cole Hopf transformation $f_N(z,\tau) = \frac{1}{N}\partial_z \ln D_N(z,\tau)$ leads to exact for any N, viscid complex Burgers equation
  $\partial_\tau f_N + f_N \partial_z f_N = -\mu \partial_{zz} f_N \qquad \mu = \frac{1}{2N}$
- Positive viscosity "smoothens" the shocks, negative is "roughening" them, triggering violent oscillations
- Note than $G(z,\tau) = \frac{1}{N}\left\langle \mathrm{Tr} \frac{1}{z-H(\tau)} \right\rangle =$ $\partial_z \left\langle \frac{1}{N} \mathrm{Tr} \ln(z-H(\tau)) \right\rangle = \partial_z \left\langle \frac{1}{N} \ln \det(z-H(\tau)) \right\rangle$ so f_N and G coincide only when $N=\infty$ (cumulant expansion).
- $\left\langle \frac{1}{N} \ln \det(z-H(\tau)) \right\rangle \stackrel{N=\infty}{=} \frac{1}{N} \ln \langle \det(z-H(\tau)) \rangle$,

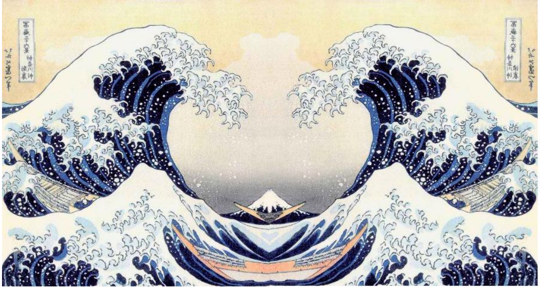
Airy function as the herald of the shock

- Shock wave corresponds to square root singularities
- Number of eigenvalues in the narrow strip of width s around branch point scales like $n = N \int_{\text{strip}} \lambda^{1/2} d\lambda = Ns^{3/2}$, so the spacing between the eigenvalues ($n = 1$) scales like $N^{-2/3} \sim \mu^{2/3}$
- Then $\pm x = 2\sqrt{\tau} + \mu^{2/3} s$ and $f_N(x, \tau) \sim \pm \frac{1}{\sqrt{\tau}} + \mu^{1/3} \xi_N(s, \tau)$
- Solving viscid Burgers equation with above parametrization yields, in the large N, limit Riccati equation, with solution $\xi_N \sim \partial_s \ln Ai(\frac{s}{2\sqrt{\tau}})$
- Herald of "soft edge" universality
- Note that despite we know in this case the exact finite viscosity solution (monic, time-dependent Hermite polynomial), we do not need its form to infer the large N asymptotics at the end-points.

Non-trivial boundary conditions

- Complex characteristics, nontrivial initial conditions
  $G(z,\tau) = G_0(\xi[z,\tau)])$ $\qquad G_0(z) = \frac{1}{2}\left(\frac{1}{z-1} + \frac{1}{z+1}\right)$
  $\xi = z - G_0(\xi)\tau$, so solution reads again
  $G(z,\tau) = G_0(z - \tau G(z,\tau))$ but now is given by the cubic (Cardano) equation
- Shock wave when $\frac{d\xi}{dz} = \infty$
- Novel phenomenon happens at $\tau = \tau^* = 1$, where square root branch points collide forming cubic root branch point (inflexion point, i.e. the change of curvature of the colliding shock waves). This triggers different scaling in viscosity (N), yielding the Pearcey function as the solution of the viscid Burgers at the collision of the shocks.

Non-trivial boundaty conditions - visualization



[Collage of Hokusai woodcut]

Two Airy heralds $Ai(x) = 2\pi \int_{-\infty}^{\infty} \exp i \left(\frac{t^3}{3} + xt \right)$ collide forming

Pearcey (Turrittin) herald $P(x, y) = \int_{-\infty}^{\infty} \exp i \left(\frac{t^4}{4} + x\frac{t^2}{2} + yt \right)$

Chiral GUE

- Temporal dynamics of the matrix $W(\tau)$

$$W(\tau) = \begin{pmatrix} 0 & K^\dagger(\tau) \\ K(\tau) & 0 \end{pmatrix}$$

  where K is a $M \times N$ complex matrix ($M > N$), whose elements are undergoing complex Brownian walk. We define "zero modes number " $\nu = M - N$ and "rectangularity number" $r = N/M$.

- We define
  $D_N^\nu(z,\tau) = \langle \det(w - W(\tau)) \rangle = w^\nu \left\langle \det(w^2 - K^\dagger K) \right\rangle$
  $\equiv z^{\nu/2} R_N^\nu(z,\tau)$, where $w^2 = z$.

- Using Grassmannian tricks we derive exact for any finite M, N evolution equations.

Chiral GUE - "cylindrical" diffusion

- Test No 1: For $M, N \to \infty$, N/M fixed, our equation for R_N^ν agrees with [Guionnet,Cabanal-Duvillard 2001] obtained in free martingale theory (complex Bru process, diffusing Marcenko-Pastur formula)
- Test No 2: For finite M, N, equations solved by $R_N^\nu = (-\tau)^N N! L_N^\nu(z/\tau)$ (time-dependent associated Laguerres)
- Main result reads
$$\partial_\tau D_N^\nu(w,\tau) = -\frac{1}{2 \cdot 2M}\frac{1}{w}\partial_w\left(w\partial_w\right)D_N^\nu(w,\tau) + \frac{1}{2 \cdot 2M}\frac{\nu^2}{w^2}D_N^\nu(w,\tau)$$
- For $\nu = 0$, this is a complex analog of the cylindrical diffusion equation with viscosity equal to the inverse of the size of the matrix (i.e. $2M$).
- CH transformation $f_{N+M} = \frac{1}{M+N}\partial_w \ln D_N^\nu(w,\tau)$ generates corresponding complex "Burgers-like" equation

Chiral GUE - three types of shocks

- In large N, M limit ($r \to 1$), but ν fixed, we again recover inviscid Burgers equation for $g(w,\tau) = \lim_{M,N\to\infty} f_{M+N}(w,\tau)$, i.e. we get $\partial_\tau g(w,\tau) + g(w,\tau)\partial_w g(w,\tau) = 0$
- For general boundary conditions $g(w,0) = g_0(w) = \frac{1}{2}\left(\frac{1}{w-1} + \frac{1}{w+1}\right)$ we get again cubic equation with three types of shock waves:
- If we define $w - w^* \equiv p$ we get three types of scalings
  1. $p \to (N+M)^{-2/3} s$ for $\tau < 1$
  2. $p \to (N+M)^{-3/4} s$ for $\tau = 1$
  3. $p \to (N+M)^{-1} s$ for $\tau > 1$

Chiral GUE - Bessel and Bessoid "heralds" of shocks

Solutions in the vicinity of shocks $\xi_N = \partial_s \ln \phi$

1. For $\tau < 1$, Airy edge, where $\phi(s) = Ai(-\sqrt{2\dot{g}_0}s)$,
2. For $\tau > 1$, Bessel edge, where $\phi(s) = s^{-\nu/2} J_\nu(\pi\rho(0)\sqrt{s})$
3. For $\tau = 1$, generalized Bessoid
   $\phi(m, r) = \int_0^\infty y^{\nu+1} e^{-y^4/2 - y^2 r} J_\nu(2my) dy$
   where variables $m = -is, r$ scale with N like $N^{3/4}, N^{1/2}$, respectively.

Note that Bessoid cusp in chiral GUE superimposes the Pearcey cusp in GUE (CUE), where no additional symmetries are imposed on the ensemble.

Morphology of singularity (Thom, Berry, Howls)

GEOMETRIC OPTICS
(wavelength $\lambda = 0$)

- trajectories: rays of light
- intensity surface: caustic

WAVE OPTICS ($\lambda \to 0$)

$N \to \infty$ Gaussian RM
($\mu = \frac{1}{2N} = 0$)

- trajectories: characteristics
- singularities of spectral flow

FINITE N (viscosity $\mu \to 0$)

Universal scaling, Arnold (α) and Berry (σ) indices

"Wave packet" scaling

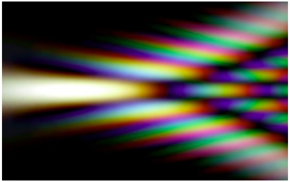
- $\Psi = \frac{C}{\lambda^\alpha}\Psi(\frac{x}{\lambda^{\sigma_x}}, \frac{y}{\lambda^{\sigma_y}})$
- fold $\alpha = \frac{1}{6}$ $\sigma = \frac{2}{3}$ Airy
- cusp $\alpha = \frac{1}{4}$ $\sigma_x = \frac{1}{2}$ $\sigma_y = \frac{3}{4}$ Pearcey

Universal scaling with N

- Zeroes of D_N
- $N^{2/3}$ scaling at the edge
- $N^{1/2}$ and $N^{3/4}$ scaling at the closure of the gap

Universal scaling visualization - "classical" analogy

Caustics, illustration from Henrik Wann Jensen

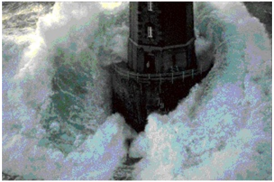


Fold and cusp fringes, illustrations by Sir Michael Berry

Diffusion of unitary matrices:

- Similar Burgers like equations for multiplicative diffusion
- Collision of two shock waves, propagating along the unit circle
- "Slow motion" documentation of CLT – Haar measure

Gapped phase
$\tau < \tau^*$



Closure of the gap
$\tau = \tau^*$

Gappless phase
$\tau > \tau^*$

Photos by Jean Guichard (La Jument lighthouse, Brittany)

Applications

- Complex systems evolve as a function of some external parameter (time, length of the wire, area of the surface, length of the box, temperature etc.)
- Lookout for universality windows where simplified dynamics of RM is shared by non-trivial theories
- Ex. 1: Strong-weak coupling transition in large N_c Yang-Mills theory (Durhuus-Olesen transition) as the shock wave collision on unitary circle [Narayanan, Neuberger, Blaizot, MAN, Lohnmayer, Wettig 2006-2012], Pearcey's critical exponents confirmed by lattice simulations in 3 and 4 dimensions.
- Ex. 2 : Chiral symmetry breakdown in Quantum Chromodynamics as the chiral shock wave collisions [Blaizot, MAN,Warchol, 2011-2012], Bessoid criticality [Janik,MAN,Papp,Zahed 1998],[Brezin,Hikami 1998]

Conclusions

- New insight for large N behavior of the random matrix models based on simple concept of "Burgulence"
- New paradigm: Characteristic polynomial is the fundamental object in RMT
- Similar structures for $P_N(z,\tau) \equiv < \frac{1}{\det(z-H(\tau))} >$, hinting deeper mathematical structures [Blaizot,MAN 2010]
- Rigid mathematical proof of the universality of the "heralds"
- Generalizations for higher dimensional shocks (non-hermitian ensembles)[Gudowska-Nowak,Janik,MAN,Jurkiewicz 2003],[Biane 1997]
- Generalizations for $\beta \neq 2$ ensembles
- Natural links to KPZ equation (growing interfaces)
- Analogies to chiral diffraction catastrophes [Berry,Jeffrey 2006]
- ...