Escape rates for Gibbs measures and conformal repellors

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1. Expanding maps of the circle K ($d = 1$);
2. Rational maps on hyperbolic Julia sets ($d = 2$).

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Let $\dim_H(K_\epsilon)$ denote the Hausdorff dimension of the set K_ϵ.
In this case it is easy to check that for any $z \in K$

$$\dim_H(K_\epsilon) \to 1 \text{ as } \epsilon \to 0.$$

It remains to understand how quickly this converges.

Asymptotic formula for the Hausdorff Dimension

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Theorem

Fix $z \in K$. Then

$$\lim_{\epsilon \to 0} \frac{1 - dim_H(K_\epsilon)}{\mu(B(z,\epsilon))} = \frac{d_\mu(z)}{\int \log |T'| d\mu}.$$

A very familiar example

Example

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1. In this case $d_\mu(z) = 1$ for all points *except* the countable set of periodic points of the form

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2. If z has prime period p then $d_\mu(z) = 1 - \frac{1}{2^p}$.

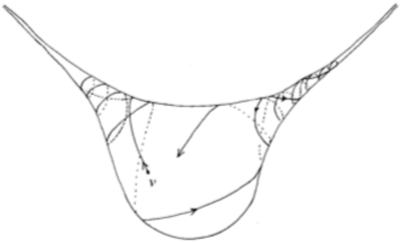
For example, the fixed point $z = 1$ has a corresponding value $d_\mu(1) = 1 - \frac{1}{2} = \frac{1}{2}$

Aside: Bounded geodesics

By way of motivation, recall a classical result of Dani.

Theorem (Dani)

The bounded geodesics on the Modular surface $M = \mathbb{H}^2/SL(2,\mathbb{Z})$, say, have full dimension $(= 3)$



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In terms of continued fractions

$$[a_1, a_2, a_3, \cdots] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$

this corresponds to the following:

$$\dim_H \left(\underbrace{\cup_{N=1}^{\infty} \{[a_1, a_2, a_3, \cdots] : a_i \leq N\}}_{\text{Bounded continued fractions}} \right) = 1.$$

Aside: Dynamical formulation using the Gauss map

Consider the usual Gauss map (or continued fraction transformation) $T : [0,1]\backslash\mathbb{Q} \to [0,1]\backslash\mathbb{Q}$ defined by

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If we denote $I_N = \left[0, \frac{1}{N}\right]$ then the corresponding result is that

$$\lim_{N\to+\infty} \dim_H \left(\left\{x \in [0,1]\backslash\mathbb{Q} : \ T^k(x) \notin I_N \text{ for all } k \geq 1\right\}\right) = 1.$$

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Moreover, there is a stronger classic asymptotic result:

Theorem (Hensley)

There exists $c > 0$ such that

$$dim_H \left(\left\{x \in [0,1] : \ T^k(x) \notin I_N \text{ for all } k \geq 0\right\}\right) = 1 - \frac{c}{N} + o\left(\frac{1}{N}\right).$$

Aside: Back to geodesics

We can reformulating the last result in terms of geodesics.

Definition

Given $T > 0$ let $\mathcal{H}(T)$ be the set of geodesics $\gamma : \mathbb{R} \to M$ for which

$$\sup_{t \geq 0}\{\log d(\gamma(0), \gamma(t))\} \leq T$$

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Theorem (after Hensley)

There exists $C > 0$ such that

$$\dim_H(\mathcal{H}(T)) = 3 - \frac{C}{T} + o\left(\frac{1}{T}\right).$$

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Let $T : X \to X$ be a continuous map. Fix $z \in X$. Let $\epsilon > 0$.

Notation

For each $n \geq 1$, we can consider the sets

$$X_{n,\epsilon} = \{x \in X \ : \ T^k(x) \notin B(z, \epsilon), \text{for all } 0 \leq k \leq n-1\},$$

i.e., those x for which the first n points in the orbit don't enter the ball.

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The basic example is the doubling map $T(z) = z^2$ and μ is the usual Haar measure on K.

This example was studied by Bunimovich and Yurchenko, and was the motivation for most of the subsequent analysis. Other examples with an absolutely continuous invariant measure were studied by Keller-Liverani.

Asymptotic formulae for escape rates

Definition

Fix $z \in K$. For $\epsilon > 0$ we define the *escape rate* of μ through $B(z, \epsilon)$ by:

$$r_\mu(B(z,\epsilon)) = -\limsup_{k\to\infty} \frac{1}{k} \log \mu\{x \in K \; : \; T^i(x) \notin B(z,\epsilon), 0 \leq i \leq k-1\}.$$

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As $\epsilon \to 0$ we have that $r_\mu(B(z,\epsilon)) \to 0$. Moreover:

Theorem

Let μ be the Gibbs measure. Then

$$\lim_{\epsilon\to 0} \frac{r_\mu(B(z,\epsilon))}{\mu(B(z,\epsilon))} = \begin{cases} 1 & \text{if } z \text{ is not periodic} \\ 1 - e^{\phi^p(z)} & \text{if } z \text{ has prime period } p \end{cases}$$

where $\phi^p(z) = \phi(z) + \phi(Tz) + \cdots + \phi(T^{p-1}z)$.

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We recover a recent result of Keller and Liverani.

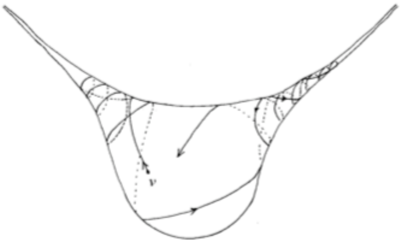
Corollary (Keller-Liverani)

Let μ be the unique T-invariant absolutely continuous probability measure, then

$$\lim_{\epsilon \to 0} \frac{r_\mu(B(z,\epsilon))}{\mu(B(z,\epsilon))} = \begin{cases} 1 & \text{if } z \text{ is not periodic} \\ 1 - 1/|(T^n)'(z)| & \text{if } z \text{ has prime period } p \end{cases}$$

Aside revisited: Sullivan's geodesic excursion theorem

There is a well known result by Sullivan on how far into the cusp typical geodesics can go in a finite time.



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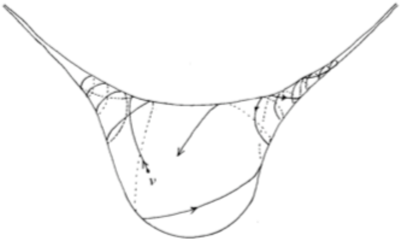

Theorem (Sullivan)

For almost every geodesic $\gamma : \mathbb{R} \to M$ we have that

$$\lim_{T \to +\infty} \frac{\sup_{0 \leq t \leq T}(d(\gamma(0), \gamma(t)))}{\log T} = 1.$$

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Theorem (Gumbel Law on the Modular surface)

For any $y > 0$ we have

$$\lim_{t \to +\infty} \mu \left\{ \dot{\gamma}(0) : \sup_{0 \leq t \leq T} (d(\gamma(t), \gamma(0))) - \log T \leq \log\left(\frac{6y}{\pi}\right) \right\} = e^{-\frac{1}{y}}.$$

Conformal repellers: The general setting

More generally, let $\mathcal{M}$ be a Riemannian manifold and $T : \mathcal{M} \to \mathcal{M}$ a C^2-map. Let J be a compact subset of $\mathcal{M}$ such that $T(J) = J$.

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2. there exists $c > 0$ and $\lambda > 1$ such that $\|DT_x^n v\| \geq c\lambda^n \|v\|$ for all $x \in J$, $v \in T_x\mathcal{M}$, and $n \geq 1$.

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3. f is topologically mixing on J.
4. J is maximal, i.e. there exists an open neighbourhood $V \supset J$ such that
$$J = \{x \in V \ : \ T^n(x) \in V \text{ for all } n \geq 0\}.$$

The natural analogues of the theorems on intervals maps still hold.

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3. Interpret the pressure as the maximal eigenvalue of a (Ruelle transfer) operator and use some messy perturbation theory to show its dependence.

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1. Use Markov partitions to model the map T symbolically;
2. Relate the Hausdorff dimension to the theormodynamic pressure; (Bowen-Ruelle)
3. Interpret the pressure as the maximal eigenvalue of a (Ruelle transfer) operator and use some messy perturbation theory to show its dependence.

It is easier to illustrate this by describing the eigenvalues of matrices.

A simplified problem: Eigenvalues of matrices

Assume that we are given a $l \times l$ matrix A ($l \geq 2$) with entries 0 or 1. Assume that the matrix is aperiodic, i.e., there exists $N > 0$ such that $A^N > 0$.

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Question

How big is the change in the maximal eigenvalue?

An Example

Let $l = 3$ and consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

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which has eigenvalue $\lambda(A) = 3$. Changing the $(1,1)$ entry to 0 gives:

$$B = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

which has smaller eigenvalue $\lambda(B) = 1 + \sqrt{3} = 2.73205\cdots$.

Powers of matrices

More generally, consider the nth fold matrix product $A^n := A.A. \cdots A$ (n-fold) which now has entries The matrix A^n has a maximal positive eigenvalue $\lambda(A^n) = \lambda(A)^n > 0$.

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Assume that we reduce one of the positive entries in A^n by 1 to get a new matrix $B^{(n)}$. (Equivalently, we forbid one of the paths of length n.)

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Let $B^{(n)} = A^n - E_{11}$, i.e., we reduce the entry in the first row and first column by 1.

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The maximal eigenvalue satisfies $\lambda(B^{(n)}) < \lambda(A^n)$.

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Example revisited

For $n \geq 1$ consider the matrices

$$A^n = \begin{pmatrix} 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \end{pmatrix}$$

which have eigenvalue $\lambda(A) = 3^n$.

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Reducing the $(1,1)$ entry by one gives:

$$B^{(n)} = \begin{pmatrix} 3^{n-1} - 1 & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \end{pmatrix}$$

Example revisited

For $n = 2, 3, 4, 5, 6, 7, \cdots$ we can compare $\lambda(A^n)$ and $\lambda(B^{(n)})$:

n	$\lambda(A^n)$	$\lambda(B^{(n)})$
2	9	$8.69042\cdots$
3	27	$26.6748\cdots$
4	81	$80.6694\cdots$
5	243	$242.668\cdots$
6	729	$728.667\cdots$
7	2187	$2186.67\cdots$

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In fact one has an estimate of Lind:

Theorem (Lind)

There exists a constant $c > 1$ such that

$$\frac{1}{c} \leq \frac{\log 3 - \frac{1}{n}\log\lambda(B^{(n)})}{(1/3)^n} \leq c \quad \text{for all } n.$$

An asymptotic version for the example

Question

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We have that

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This extends to subshifts of finite type, and transfer operators.

Subshifts of finite type

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Let A (again) denote an aperiodic $l \times l$ matrix of zeroes and ones, i.e. for each $1 \leq i, j \leq l$, there exists a positive integer N such that $A^N(i,j) = 1$.

Definition

We define the *subshift of finite type* on the space

$$\Sigma = \{x = (x_k)_{k=0}^{\infty} : A(x_k, x_{k+1}) = 1, \text{ for all } k\}$$

to be the map $\sigma : \Sigma \to \Sigma$ by $(\sigma x)_k = x_{k+1}$.

Orbits that avoid cylinders

Definition

Given $z = (z_k)_{k=0}^{\infty} \in \Sigma$ and $n \geq 0$ we can associate a *cylinder set*

$$[z]_n = \{x = (x_k)_{k=0}^{\infty} : x_k = z_k \text{ for } k = 0, \cdots, n-1\}.$$

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Example

$h(\sigma) = \log \lambda(A)$ and $h(\sigma|\Sigma_n) = \frac{1}{n} \log \lambda(B^{(n)})$

First order entropy change

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Let μ be the measure of maximal entropy for $\sigma : \Sigma \to \Sigma$ (i.e., the Parry measure) then we have the following asymptotic estimate:

Theorem

We have that

$$\lim_{n\to\infty} \frac{h(\sigma) - h(\sigma|\Sigma_n))}{\mu([z]_n)} = \begin{cases} 1 & \text{if } z \text{ is not periodic} \\ 1 - e^{-ph(\sigma)} & \text{if } z \text{ has prime period } p. \end{cases}$$

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Example

The Parry measure is the Bernoulli measure $\mu = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)^{\mathbb{Z}_+}$