Burkholder functionals, Morrey's question and singular integrals

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Joint work with

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Consider the variational problem

$$\min \int_{\Omega} \mathcal{F}\Big(D\phi(x)\Big).$$

One looks for a minimizer over a suitable subclass of maps

$$\phi : \mathbb{R}^n \to \mathbb{R}^m.$$

Above the integrand $\mathcal{F}$ depends only on the pointwise values of the gradient $D\phi(x) \in \mathbb{R}^{n\times m}$ (and $\Omega \subset \mathbb{R}^n$ is a given domain).

A very useful property of the functional is the (weak) lower semicontinuity:

$$\int_\Omega \mathcal{F}\Big(D\phi(x)\Big) \leqslant \liminf_{j\to\infty} \int_\Omega \mathcal{F}\Big(D\phi_j(x)\Big) \quad \text{whenever} \quad \phi_j \xrightarrow{w^*} \phi \text{ in } W^{1,\infty}(\Omega, \mathbf{R}^m).$$

When does this hold?

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Morrey (1952): Lower semicontinuity is equivalent to $\mathcal{F} : \mathbb{R}^{n\times m} \to \mathbb{R}$ being *quasiconvex*.

$\mathcal{F}$ is quasiconvex if all linear functions $A : \mathbb{R}^n \to \mathbb{R}^m$ provide local minima with respect to compact perturbations, i.e.

$$\int_\Omega \mathcal{F}(A) \leqslant \int_\Omega \mathcal{F}(A + Dh(x)), \quad \text{for all } h \in C_0^\infty(\Omega, \mathbb{R}^m).$$

(the notion is independent of Ω)

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Classical: convexity of $\mathcal{F}$ implies quasiconvexity. Moreover,

$n = 1$: quasiconvexity $\Leftrightarrow$ $\mathcal{F}$ convex.

A necessary condition for quasiconvexity is that $\mathcal{F}$ is rank-one convex

Definition: $\mathcal{F}$ is rank-one convex if it is convex to all rank-one directions , i.e.

$$\mathbb{R} \ni t \mapsto \mathcal{F}(A + tB) \quad \text{is convex}$$

for all $A, B \in \mathbb{R}^{n \times m}$ with $\text{rank}(B) = 1$.

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One defines quasiconcavity in an analoguous fashion ($\leq$ is replaced by $\geq$) and again rank-one concavity is a necessary condition.

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Question of Morrey: Is rank-one convexity also sufficient for quasiconvexity??

Sverak: No, if $n \geqslant 3$. Kristensen: no local characterization ($n \geqslant 3$).

Since Sverak's counterexample works only in dimension $n \geq 3$, we may state an

Updated version of Morrey's question:

Assume that the continuous $\mathcal{F} : \mathbb{R}^{2\times 2} \to \mathbb{R}$ is *rank-one convex*. Does this imply that $\mathcal{F}$ is *quasiconvex* ?

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$*$ positive answer would have direct interesting implications

$*$ J. Ball: polyconvexity yields a sufficient condition.

Examples.

1. $$\mathbf{MIN}\left(\int_0^1 \left((\phi'(x))^4 + (\phi'(x))^2\right) dx\right) \quad \text{under} \quad \begin{cases} \phi(0) = a, \\ \phi(1) = b. \end{cases}$$

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2. $$\text{MIN}\left(\int_Q \Big(((\phi_x)^2 + (\phi_y)^2)^2 + (J_\phi)^2\,dx\right) \quad \text{under} \quad \phi_{|\partial Q} = g,$$

where $Q = [0,1]^2$.

Functional is polyconvex (i.e. convex function of $D\phi$ and its minors) $\Rightarrow$ quasiconvex. As before, in a suitable function space the existence of a minimizer is guaranteed.

Examples.

3. $$\mathbf{MIN} \left(\int_0^1 (\phi'(x) - 1)^2 (\phi'(x) + 1)^2 \, dx \right) \quad \text{under} \quad \begin{cases} \phi(0) = 0, \\ \phi(1) = 0. \end{cases}$$

Functional is not quasiconvex $\Rightarrow$ weak lower semicontinuity fails!
However, in this case minimizer exists, but is far from unique.

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3. $$\textbf{MIN}\left(\int_0^1 (\phi'(x)-1)^2(\phi'(x)+1)^2\,dx\right) \quad \text{under} \quad \begin{cases} \phi(0)=0, \\ \phi(1)=1. \end{cases}$$

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4. $$\textbf{MIN}\left(\int_Q \Big(((\phi_x)^2-1)^2+(\phi_y)^4\,dx\right) \quad \text{under } \phi_{|\partial Q}=0, \quad \text{where}$$

$Q=[0,1]^2$.

Functional is not quasiconvex $\Rightarrow$ weak lower semiconinuity fails again.
In this case minimizers do not exist, although the functional is coercive and nice in many ways!

Burkholder functional:

Assume $p \geq 1$. For $f \in C^\infty(\mathbb{R}^2, \mathbb{R}^2)$, define the Burkholder functional by

$$\mathcal{B}_p\Big(Df(x)\Big) = \Big(\, p\, J(x, f) \,+ (2-p)\, |Df(x)|^2\Big) \cdot |Df(x)|^{p-2}$$

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- Conjecture: if $p \geq 2$, then the Burkholder functional $\mathcal{B}_p$ is quasiconcave. Positive answer $\Longrightarrow$ striking consequences, like an affirmative answer to the Iwaniec conjecture.

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- Let (X_k) and (Y_k) are be martingales with differential subordination: $|Y_k - Y_{k-1}| \leq |X_k - X_{k-1}|$ a.s.. Then if $p^* := \max(p, p/(p-1))$, one has

$$\|(Y_k)\|_p := \sup_{k\geq 1} \|Y_k\|_{L^p(\Omega)} \leq (p^* - 1)\|(X_k)\|_p.$$

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Various generalizations by Burkholder, Banuelos, ...

• A beautiful corollary: The unconditionality constant of the Haar basis on the space $L^p(0,1)$ equals $p^* - 1$.

Applications to singular integrals

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- Results often in sharp (or best known) estimates.

- Lot of interesting work done by Banuelos and his collaborators. Volberg, Nazarov, Treil, Petermichl and their collaborators have applied Bellman function techniques with great success on similar problems.

Example: a stochastic representation of a singular integral – the real part of the Beurling-Ahlfors transform S:

$$\operatorname{Re} Sf(x) = \lim_{T\to\infty} 2\pi T\, \mathbb{E}\left[\int_0^T \nabla u(T-t, W_{T-t}) \cdot A dW_t\right]$$

Here

– $\operatorname{Re} S := R_1^2 - R_2^2$, where R_j is the j:th Riesz tranform: $\widehat{(R_j f)} = -i(\xi_j/|\xi|)\widehat{f}$,

– u is the heat extension of the function f to the upper half space $\mathbb{R}^3_+$,

–W_t is 2-d Brownian motion started at origin at $t = 0$, conditioned on $W_T = x$,

–A is the matrix $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Beurling-Ahlfors operator: $$Sf(z) := -\frac{1}{\pi}\int_{\mathbb{R}^2} \frac{f(w)dA(w)}{(z-w)^2}$$

$$= \Big((\mathrm{Re}\, S + i\mathrm{Im}\, S)f\Big)(z) = \Big((R_1^2 - R_2^2 + i2R_1R_2)f\Big)(z).$$

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- Best known upper bound today is due to Banuelos and Janakiraman (2007): $\|S\|_p \le 1.575(p-1)$
- A surprising lower bound: $\|\mathrm{Re}\, S\|_p \ge p-1 \Rightarrow \|\mathrm{Re}\, S\|_p = p-1.$
(Geiss, Mongomery-Smith and S. (2010)). Based on adapting techniques of Bourgain from theory of martingales with values in Banach spaces with UMD property.

Back to quasiconvexity properties of the Burkholder functional. Our (A & I & P & S) main result to this direction is the following:

Theorem 1. Let $\Omega \subset R^2$ be a bounded domain. Assume $f : \Omega \to \mathbb{R}^2$, with

1. $f - Id \in \mathbb{C}_0^\infty(\Omega)$, and
2. $\mathcal{B}_p\Big(Df(x)\Big) \geq 0$ pointwise in Ω.

Then

$$\int_\Omega \mathcal{B}_p\,(Df)\,dx \leqslant \int_\Omega \mathcal{B}_p\,(Id\,)\,dx = 2|\Omega|, \qquad p \geqslant 2.$$

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1. $f - Id \in \mathbb{C}_0^\infty(\Omega)$, and
2. $|Df(x)|^2 \leqslant \frac{p}{p-2} J(x, f)$ pointwise in Ω ($\Rightarrow$ qc-maps)

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Extremals.

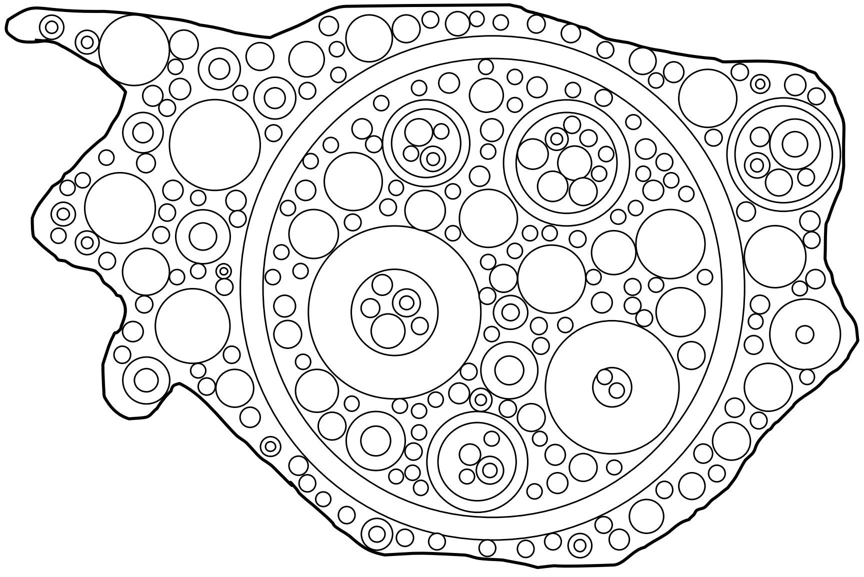
Consider radial maps $f(z) = \frac{z}{|z|}\,\rho(|z|)$, $\rho : [0,1] \to [0,1]$ homeo.

If $\rho(t) \geqslant t\,\rho'(t) > 0$,

$$\int_{\mathbb{D}} \mathcal{B}_p\Big(Df(x)\Big) = \int_{\mathbb{D}} \mathcal{B}_p(\,Id\,), \qquad p \geqslant 1.$$

In arbitrary domains: more complicated exaples by disk fillings !

Hence there is a rich class of extremals!



Extremals with fractal like structure

All these are local maxima for Burkholder integrals !

Corollary 1. Every homeomorphism $f : \Omega \longrightarrow \Omega$, with

$$f(z) - z \in W_0^{1,2}(\Omega),$$

satisfies

$$\int_\Omega \Big(1 + \log |Df(z)|^2\Big)\, J(z,f)\, \mathrm{d}z \;\leqslant\; \int_\Omega |Df(z)|^2\, \mathrm{d}z$$

Equality occurs for the identity map,
as well as for a number of piece-wise radial mappings.

Corollary 2. For the Beurling-Ahlfors operator

$$(\mathcal{S}\phi)(z) = -\frac{1}{\pi} \iint_{\mathbb{C}} \frac{\phi(\tau)}{(z-\tau)^2}\, d\tau$$

and for any function with $|\mu(z)| \leq 1, \quad z \in \mathbb{D},$ we have

$$\int_{\mathbb{D}} \Big(1 \; - \; |\mu(z)|\Big)\, e^{\mathrm{Re}\, S\mu(z) + |\mu|} dz \leq \pi.$$

Equality occurs for:

$$\mu(z) = -\frac{z}{\bar{z}} a(|z|)\chi_{\mathbb{D}}(z), \qquad 0 \leqslant a(t) \leqslant 1, \quad \int_0 \Big(1 - a(t)\Big)\frac{dt}{t} = +\infty$$

Corollary 3. for quasiconformal mappings: Suppose

- $f \in W^{1,2}_{loc}(\Omega)$-homeo, with $|Df(x)| \leqslant KJ(x,f)$ for a.e. $x \in \Omega$.
- $f(x) = x$ for $x \in \partial\Omega$.

Then, writing $K(x) = |Df(x)|/J(x,f)$, one has

$$\frac{1}{|\Omega|}\int_\Omega \left(\frac{1}{K} - \frac{1}{K(x)}\right) \left|Df(z)\right|^{2K/(K-1)} \, \mathrm{d}z < \infty \ .$$

Proof of Theorem 1

Through new kind of interpolation of analytic families

Interpolation Lemma. Let $0 < p_0,\ p_1 \leqslant \infty$, $\quad \mathbb{H}_+ = \{\lambda \in \mathbb{C} : \text{Re}\lambda > 0\}$

If $\{\phi_\lambda = \phi_\lambda(x)\,;\ \lambda \in \mathbb{H}_+\}$, analytic family of meas. functions, with

$$M_1 := \|\,\phi_1\|_{p_1} < \infty, \qquad M_0 := \sup_{\lambda \in \mathbb{H}_+} \|\,\phi_\lambda\|_{p_0} < \infty,$$

Then: $$M_\theta \leqslant M_0^{1-\theta} \cdot M_1^{\theta} < \infty\,, \qquad 0 < \theta < 1\,,$$

where $M_\theta := \left\|\phi_\theta\right\|_{p_\theta}$ with $\quad \dfrac{1}{p_\theta} = (1-\theta)\cdot\dfrac{1}{p_0} + \theta\cdot\dfrac{1}{p_1},$

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Too good to be true !

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If $\{\phi_\lambda = \phi_\lambda(x)\,;\ \lambda \in \mathbb{H}_+\}$, analytic family of meas. functions, with

$$\phi_\lambda(x) \neq 0, \quad \text{a.e. } x \in \Omega,\ \lambda \in \mathbb{H}_+,$$

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Proof of the Interpolation Lemma is not based on subharmonicity. Instead one applies thermodynamics related inequalities.

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- Write the Theorem as an inequality for $(p-1)$-quasiconformal maps.

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- Make sure to get right L^∞-bound at $\lambda = 0$ and L^2-bound for all values of λ. Finally, apply a version of the interpolation lemma in the unit disc with a suitably chosen measure space.

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- A difficulty: The choice of the family ϕ_λ is quite tricky as one wants sharp estimates.

THANKS !