Sets which are not tube null and projections of random measures

Pablo Shmerkin, joint work with Ville Suomala (Oulu)

Department of Mathematics
University of Surrey

Porquerolles, 13 Juin 2011

Outline

1. Tube null sets and localization
   - Tube-null sets
   - Localization of the Fourier transform
   - Results on non-tube-null sets

2. Projections of random measures
   - Projections of measures and tube-null sets
   - Fractal percolation and projections
   - Projections of more general random measures

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Setting

- We work in $\mathbb{R}^2$ for simplicity. Everything I'll say works in any dimension.
- Everything I discuss takes places in some nice bounded domain $\Omega \subset \mathbb{R}^2$, for example the unit disk or unit square.
- Lebesgue measure on Ω is denoted $\mathcal{L}$.
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Tube-null sets

Definition

A set $E \subset \Omega$ is tube-null if, for any $\varepsilon > 0$, it can be covered by a countable union of tubes $\{T_i\}$ with $\sum_i \mathcal{L}(T_i) < \varepsilon$.

Easy Remarks

- *Any tube-null set is null.*
- *A subset of a tube-null set is tube-null.*
- *If πE is null (in $\mathbb{R}$) for some linear $\pi : \mathbb{R}^2 \to \mathbb{R}$, then E is tube-null.*

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Properties of tube-null sets

- (Carbery-Soria-Vargas) If E has σ-finite $\mathcal{H}^1$-measure, then E is tube-null (idea: decompose E into rectifiable and purely unrectifiable parts).
- (Harangi) The Von Koch snowflake is tube-null.
- (Carbery-Soria-Vargas) Let $C \subset [0,1]$ be a Cantor set of dimension $> 1/2$. The Cantor target

$$E = \{re^{2\pi it} : r \in C, t \in [0,1]\}$$

is not tube-null (Note: $\dim_H(E) \geq 1 + \dim_H(E) > 3/2$).

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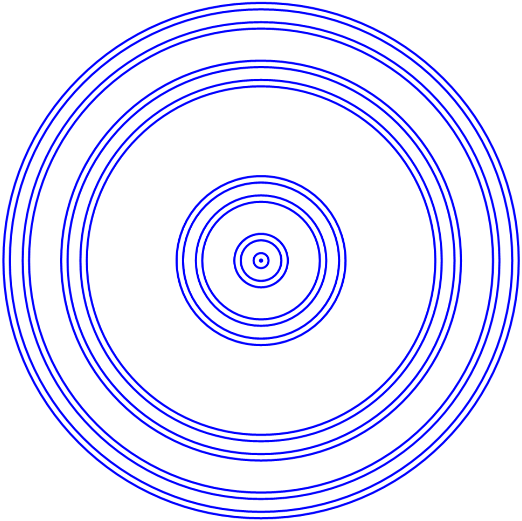
A non-tube-null Cantor target (dim $> 3/2$)



The localization problem

Definition

Given $f \in L^2(\mathbb{R}^d)$, let

$$S_R f(x) = \int_{|\xi|<R} \widehat{f}(\xi) 2^{2\pi i x\cdot\xi} d\xi$$

be the localization of f to frequencies of modulus $\leq R$.

Open problem

Is is true that for any $f \in L^2$,

$$f(x) = \lim_{R\to\infty} S_R f(x) \quad \text{for almost every } x\ ?.$$

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Famous result in dimension 1. Open in higher dimensions.

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Let Ω be a compact domain (for example unit disk). If $f \in L^2(\mathbb{R}^2)$ and supp$(f) \cap \Omega = \emptyset$, then

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Questions on non-tube-null sets

Motivated by this connection, Carbery et al asked questions on the structure of non-tube-null sets:

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- *What is* $\inf\{\dim_H(E) : E$ *is not tube null*$\}$*? (they show it is between* 1 *an* $3/2$*).*
- *For which s are there sets with* $0 < \mathcal{H}^s(E) < \infty$ *which are not tube null?*
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Answers on non-tube-null sets

Theorem (P.S. and V.Suomala 2010)

Let h be a gauge function with $h(2r) \leq 4h(r)$ and $h(r) < r/|\log r|^{3+\delta}$ for some $\delta > 0$. Then there exists a purely unrectifiable set $E \subset \Omega$ which is not tube-null and satisfies $0 < \mathcal{H}^h(E) < \infty$.

Corollary (Answers to Carbery et al's questions)

- *There are non-tube-null sets of dimension 1 (and therefore of any dimension between 1 and 2).*
- *There are non-tube-null of positive and finite $\mathcal{H}^s$-measure for all $s \in (1, 2]$ (this is immediate from the previous point).*
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Bounded projections implies tube-nullity

Lemma

Let $E \subset \Omega$. Suppose there is a measure μ on Ω such that:

- $\mu(E) > 0$.
- *There is $C > 0$ such that for all lines $\ell \subset \mathbb{R}$, $P_\ell\mu$ is uniformly continuous, and its density f_ℓ satisfies $|f_\ell|_\infty < C$.*

Then E is not tube-null.

Proof.

Clearly $\mu(T) \le C\mathcal{L}(T)$ for every tube. Therefore for any collection $\{T_i\}$ covering A,

$$0 < \mu(E) \le \sum_i \mu(T_i) \le C \sum_i \mathcal{L}(T_i),$$

whence $\sum_i \mathcal{L}(T_i)$ is bounded below, and E is not tube null. □

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Projections of random sets and measures

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We work in the unit square $Q_0 = [0,1]^2$. Fix a base $M \geq 2$ and a probability $p \in (0,1)$. The fractal percolation limit set E is constructed as follows:

- *Divide Q_0 into M^2 M-adic squares. Label each of them blue with probability p and red with probability $1-p$. Discard the red ones and call E_1 the union of the blue ones.*
- *Inside each of the surviving blue squares, repeat the process, independently from the previous stage and from each other. Call the resulting union of blue squares of second stage E_2.*
- *Continue inductively to construct a nested sequence $\{E_n\}$. The limit set is $E = \bigcap_{n=1}^{\infty} E_n$.*

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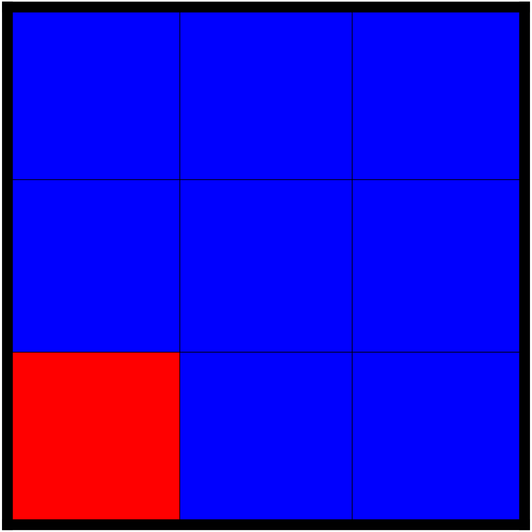

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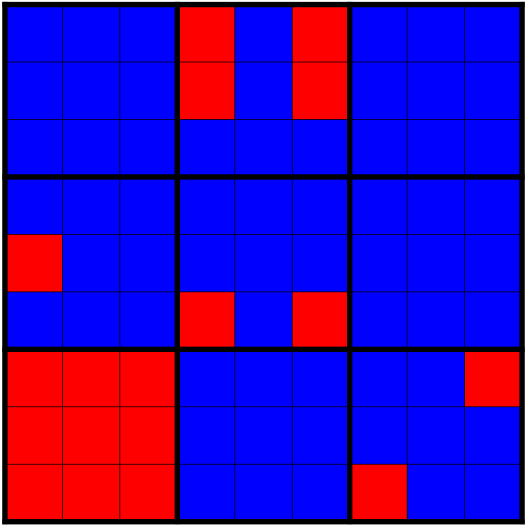

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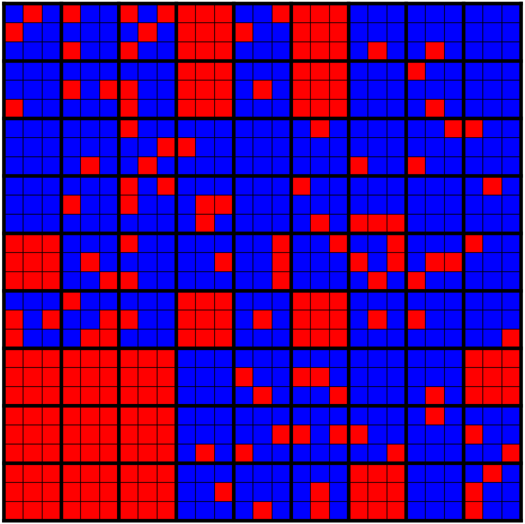

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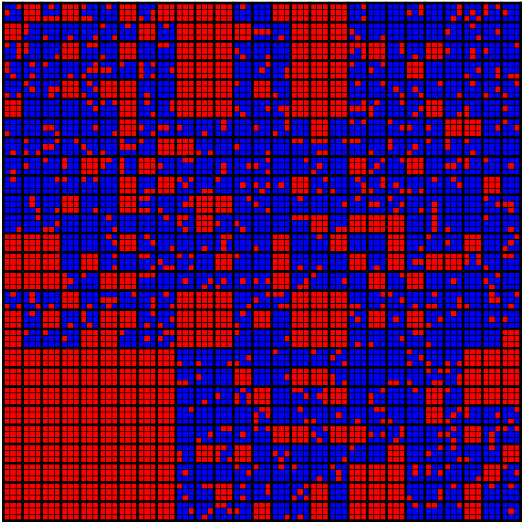

Properties of fractal percolation

- Let $s = pM^2$ be the expect number of chosen (blue) squares. Then E is almost surely empty if $s \leq 1$, but has positive probability of being nonempty if $s > 1$. Moreover, if $s > 1$ then almost surely on $E \neq \emptyset$,
$$\dim_H(E) = \dim_B(E) = \frac{\log s}{\log M}.$$
- There is $0 < p_c(M) < 1$ such that if $p < p_c$ then a.s. E is totally disconnected, and if $p > p_c$ then with positive probability E contains a connected component connecting the left and right sides of the unit square ("E percolates").

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Projections of fractal percolation sets

Theorem (M.Rams and K. Simon 2010)

Suppose $p > 1/M$ *(so that* $\dim_H(E) > 1$ *a.s.). Then almost surely, all orthogonal and also all radial projections of* E *have nonempty interior.*

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Fractal percolation measure

Definition

Recall that E_n are the stages in the construction of E. Write $\mu_n = p^n \mathcal{L}|_{E_n}$. Then a.s. μ_n converges weakly to some random measure μ supported on E. We call μ the natural fractal percolation measure.

Theorem (Y. Peres and M. Rams 2011)

Fix $p \in (1/M, 1)$ (so that $\dim E > 1$ *a.s. on $E \neq \varnothing$).*

- *There is $\gamma > 0$ such that almost surely $\pi_\ell \mu$ is Hölder continuous with exponent γ for all non vertical/horizontal lines ℓ.*
- *The constant remains bounded as long as ℓ is a positive distance away from being vertical/horizontal.*
- *If ℓ is the x or y-axis, then $\pi_\ell \mu$ is still absolutely continuous with a bounded (but discontinuous) density.*

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Theorem (P.S. and V. Suomala 2011)

We give some abstract conditions on a random measure μ on Ω which guarantee that a.s. there is C (random) such that for every line ℓ, $\|P_\ell \mu\| < C$. Very roughly, the conditions are:

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- *Given μ_n, the μ_{n+1}-measure of sets which are far away are independent random variables.*
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- The natural measure on fractal percolation is one example of a measure satisfying our conditions, but there are many others.
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Uniform continuity of projections

- We recall that Rams and Peres proved that, other than exceptional directions, projections of the fractal percolation measure are continuous with a Hölder density. It is natural to ask if there are fractal measures all of whose projections are continuous and, in this case, how smooth they can be.
- Somewhat related to this, fractal percolation has undesirable exceptional directions. Since the results we seek are rotation invariant, perhaps random measures with rotational invariance would be more appropriate.
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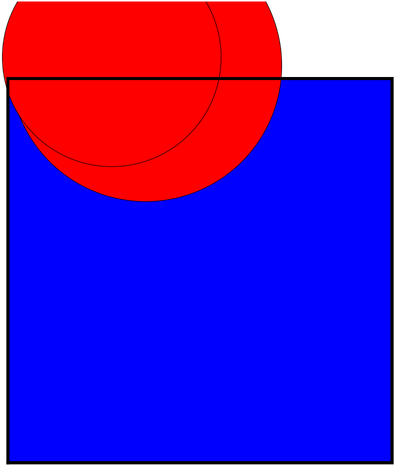

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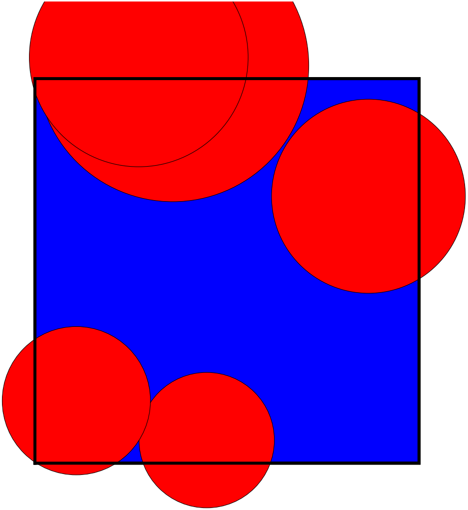

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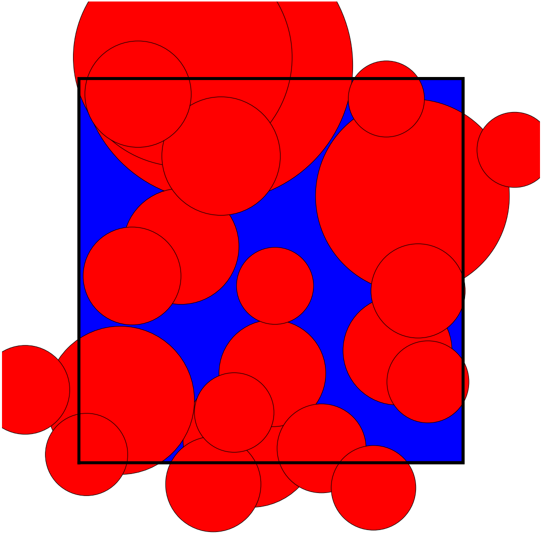

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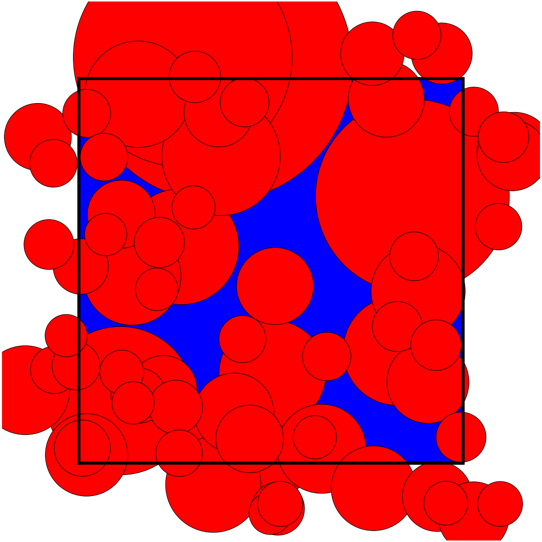

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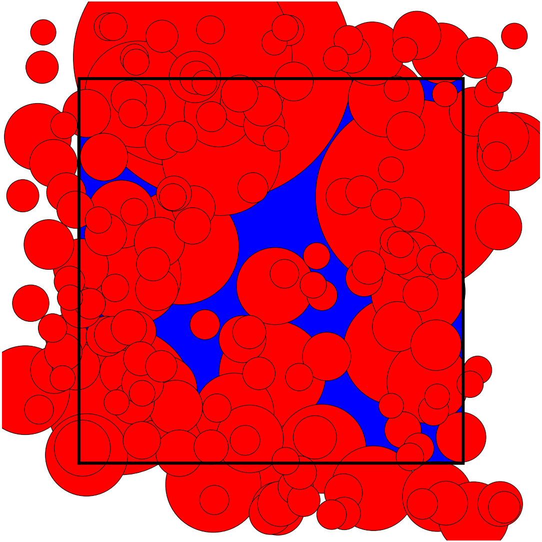

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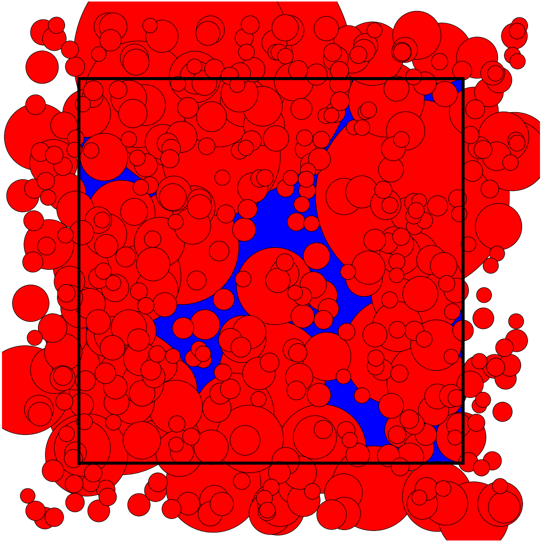

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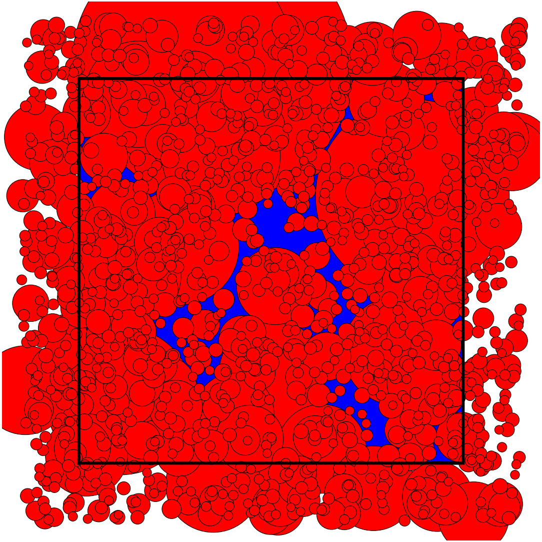

A class of examples: Poissonian cutout measures

- Write $\mathcal{X}$ =compact subsets of $\mathbb{R}^2$, with Hausdorff metric.
- Let $\mathbf{Q}$ be a measure on $\mathcal{X}$ which is translation invariant, scale invariant and locally finite (this means that the measure of all subsets of $[-1,1]^2$ with diameter > 1 is finite).
- Let $\mathcal{Y} = \{\Lambda_j\}$ be a Poisson point process with intensity $\mathbf{Q}$. This means $\mathcal{Y}$ is a random countable collection of compact sets such that if $\{\mathcal{X}_i\}$ are pairwise disjoint subsets of $\mathcal{X}$, then the random variables $\#(\mathcal{Y} \cap \mathcal{X}_i)$ are independent Poisson distributed with mean $\mathbf{Q}(\mathcal{X}_i)$.
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Open questions and thanks

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