Sets which are not tube null and projections of random measures

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Porquerolles, 13 Juin 2011
Outline

1 Tube null sets and localization
   - Tube-null sets
   - Localization of the Fourier transform
   - Results on non-tube-null sets

2 Projections of random measures
   - Projections of measures and tube-null sets
   - Fractal percolation and projections
   - Projections of more general random measures
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   - Projections of more general random measures
Setting

- We work in $\mathbb{R}^2$ for simplicity. Everything I’ll say works in any dimension.
- Everything I discuss takes places in some nice bounded domain $\Omega \subset \mathbb{R}^2$, for example the unit disk or unit square.
- Lebesgue measure on $\Omega$ is denoted $\mathcal{L}$.
- An $\varepsilon$-tube $T$ is an $\varepsilon$-neighborhood of a line (intersected with $\Omega$!). It has area $O(\varepsilon)$.
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**Tube-null sets**

**Definition**

A set $E \subset \Omega$ is **tube-null** if, for any $\varepsilon > 0$, it can be covered by a countable union of tubes $\{T_i\}$ with $\sum_i \mathcal{L}(T_i) < \varepsilon$.

**Easy Remarks**

- Any tube-null set is null.
- A subset of a tube-null set is tube-null.
- If $\pi E$ is null (in $\mathbb{R}$) for some linear $\pi : \mathbb{R}^2 \to \mathbb{R}$, then $E$ is tube-null.
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Properties of tube-null sets

(Carbery-Soria-Vargas) If $E$ has $\sigma$-finite $\mathcal{H}^1$-measure, then $E$ is tube-null (idea: decompose $E$ into rectifiable and purely unrectifiable parts).

(Harangi) The Von Koch snowflake is tube-null.

(Carbery-Soria-Vargas) Let $C \subset [0, 1]$ be a Cantor set of dimension $> 1/2$. The Cantor target

$$E = \{re^{2\pi it} : r \in C, t \in [0, 1]\}$$

is not tube-null (Note: $\dim_H(E) \geq 1 + \dim_H(E) > 3/2$).
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A non-tube-null Cantor target (dim $> 3/2$)
The localization problem

**Definition**

Given $f \in L^2(\mathbb{R}^d)$, let

$$S_R f(x) = \int_{|\xi| < R} \hat{f}(\xi) 2^{2\pi i x \cdot \xi} d\xi$$

be the localization of $f$ to frequencies of modulus $\leq R$.

**Open problem**

Is it true that for any $f \in L^2$, $f(x) = \lim_{R \to \infty} S_R f(x)$ for almost every $x$?

**Remark**

Famous result in dimension 1. Open in higher dimensions.
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Partial results on localization

**Theorem (Carbery-Soria 1988)**

Let $\Omega$ be a compact domain (for example unit disk). If $f \in L^2(\mathbb{R}^2)$ and $\text{supp}(f) \cap \Omega = \emptyset$, then

$$S_R f(x) \to 0 \quad \text{for almost every } x \in \Omega.$$ 

**Remark**

In dimension 1, this is true with uniform (everywhere) convergence (Riemann localization principle).
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SDLP’s

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A set $E \subset \Omega$ is called a Set of divergence for the localization problem (SDLP) if there exists $f \in L^2(\mathbb{R}^2)$ with $\text{supp}(f) \cap \Omega = \emptyset$, such that

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Motivated by this connection, Carbery et al asked questions on the structure of non-tube-null sets:

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- What is $\inf \{ \dim_H(E) : E \text{ is not tube null} \}$? (they show it is between 1 and 3/2).

- For which $s$ are there sets with $0 < \mathcal{H}^s(E) < \infty$ which are not tube null?

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Let \( h \) be a gauge function with \( h(2r) \leq 4h(r) \) and \( h(r) < \frac{r}{|\log r|^{3+\delta}} \) for some \( \delta > 0 \). Then there exists a purely unrectifiable set \( E \subset \Omega \) which is not tube-null and satisfies \( 0 < \mathcal{H}^h(E) < \infty \).

Corollary (Answers to Carbery et al’s questions)

- There are non-tube-null sets of dimension 1 (and therefore of any dimension between 1 and 2).
- There are non-tube-null sets of positive and finite \( \mathcal{H}^s \)-measure for all \( s \in (1, 2] \) (this is immediate from the previous point).
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Lemma

Let \( E \subset \Omega \). Suppose there is a measure \( \mu \) on \( \Omega \) such that:

1. \( \mu(E) > 0 \).
2. There is \( C > 0 \) such that for all lines \( \ell \subset \mathbb{R} \), \( P_\ell \mu \) is uniformly continuous, and its density \( f_\ell \) satisfies \( |f_\ell|_\infty < C \).

Then \( E \) is not tube-null.

Proof.

Clearly \( \mu(T) \leq C \mathcal{L}(T) \) for every tube. Therefore for any collection \( \{ T_i \} \) covering \( A \),

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In the last few years, there has been much interest in proving that for different classes of random sets, certain/all projections contain intervals.

If a set $E \subset \Omega$ has the property that $\pi_\ell(E)$ contains (or is) an interval for all $\ell$, then $E$ is a good candidate to being tube-null. But we really need a measure on it to prove it is tube null.

Projections of random fractal measures have not been studied as much. But there is a recent result of Y. Peres and M. Rams which almost answers the questions on non-tube-null sets (but not quite).
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Fractal percolation

**Definition**

*We work in the unit square* \( Q_0 = [0, 1]^2 \). *Fix a base* \( M \geq 2 \) *and a probability* \( p \in (0, 1) \). *The fractal percolation limit set* \( E \) *is constructed as follows:*

- *Divide* \( Q_0 \) *into* \( M^2 \) *\( M \)-adic squares. Label each of them blue with probability* \( p \) *and red with probability* \( 1 - p \). *Discard the red ones and call* \( E_1 \) *the union of the blue ones.*

- *Inside each of the surviving blue squares, repeat the process, independently from the previous stage and from each other. Call the resulting union of blue squares of second stage* \( E_2 \).*

- *Continue inductively to construct a nested sequence* \( \{ E_n \} \). *The limit set is* \( E = \bigcap_{n=1}^{\infty} E_n \).
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- Inside each of the surviving \textbf{blue} squares, repeat the process, independently from the previous stage and from each other. Call the resulting union of \textbf{blue} squares of second stage $E_2$.

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Fractal percolation with $M = 3$, $p = .85$ (dim 1.852)
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Properties of fractal percolation

- Let $s = pM^2$ be the expect number of chosen (blue) squares. Then $E$ is almost surely empty if $s \leq 1$, but has positive probability of being nonempty if $s > 1$. Moreover, if $s > 1$ then almost surely on $E \neq \emptyset$,

\[
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- There is $0 < p_c(M) < 1$ such that if $p < p_c$ then a.s. $E$ is totally disconnected, and if $p > p_c$ then with positive probability $E$ contains a connected component connecting the left and right sides of the unit square (“$E$ percolates”).
Properties of fractal percolation

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Projections of fractal percolation sets

Theorem (M. Rams and K. Simon 2010)

Suppose \( p > 1/M \) (so that \( \dim_H(E) > 1 \) a.s.). Then almost surely, all orthogonal and also all radial projections of \( E \) have nonempty interior.

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Fractal percolation measure

**Definition**

*Recall that $E_n$ are the stages in the construction of $E$. Write $\mu_n = p^n \mathcal{L}|_{E_n}$. Then a.s. $\mu_n$ converges weakly to some random measure $\mu$ supported on $E$. We call $\mu$ the **natural fractal percolation measure**.*

**Theorem (Y. Peres and M. Rams 2011)**

*Fix $p \in (1/M, 1)$ (so that $\dim E > 1$ a.s. on $E \neq \emptyset$).

- There is $\gamma > 0$ such that almost surely $\pi_{\ell} \mu$ is Hölder continuous with exponent $\gamma$ for all non vertical/horizontal lines $\ell$.
- The constant remains bounded as long as $\ell$ is a positive distance away from being vertical/horizontal.
- If $\ell$ is the $x$ or $y$-axis, then $\pi_{\ell} \mu$ is still absolutely continuous with a bounded (but discontinuous) density.*
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Theorem (P.S. and V. Suomala 2011)

We give some abstract conditions on a random measure $\mu$ on $\Omega$ which guarantee that a.s. there is $C$ (random) such that for every line $\ell$, $\|P_\ell \mu\| < C$. Very roughly, the conditions are:

- $\mu = \lim_n \mu_n$, where $\mu_n$ is absolutely continuous ($\mu_n$ is the $(2^{-n})$-approximation to $\mu$).
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The natural measure on fractal percolation is one example of a measure satisfying our conditions, but there are many others. Applying the result to a suitable variant of fractal percolation, we obtain the examples of non-tube null sets mentioned earlier. In some sense, the generality of the result says that non-tube null sets are not really exceptional.
Remarks on main result 1

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Uniform continuity of projections

- We recall that Rams and Peres proved that, other than exceptional directions, projections of the fractal percolation measure are continuous with a Hölder density. It is natural to ask if there are fractal measures all of whose projections are continuous and, in this case, how smooth they can be.

- Somewhat related to this, fractal percolation has undesirable exceptional directions. Since the results we seek are rotation invariant, perhaps random measures with rotational invariance would be more appropriate.

- A Fourier argument shows that if $\mu$ is a measure in $\mathbb{R}^2$ of dimension $< 2$, then it is not possible for all projections to be uniformly Lipschitz. So the best we can expect is Hölder continuity.
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There are examples of such measures of any dimension larger than 1 which are purely unrectifiable.
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A class of examples: Poissonian cutout measures

- Write $\mathcal{X}$ = compact subsets of $\mathbb{R}^2$, with Hausdorff metric.
- Let $Q$ be a measure on $\mathcal{X}$ which is translation invariant, scale invariant and locally finite (this means that the measure of all subsets of $[-1,1]^2$ with diameter $>1$ is finite).
- Let $\mathcal{Y} = \{\Lambda_j\}$ be a Poisson point process with intensity $Q$. This means $\mathcal{Y}$ is a random countable collection of compact sets such that if $\{X_i\}$ are pairwise disjoint subsets of $\mathcal{X}$, then the random variables $\#(\mathcal{Y} \cap X_i)$ are independent Poisson distributed with mean $Q(X_i)$.
- The object of interest is the cutout set $E = \Omega \setminus \bigcup_i \Lambda_i$, and the natural measure $\mu$ supported on it.
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Suppose $Q$ is as above, and in addition:

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By considering the family \( \{ tQ : t > 0 \} \) we can achieve all possible dimensions of the measure.

Start with either a disk or a Von Koch snowflake. Translate it randomly (according to Lebesgue) and scale it randomly (with density \( t^{-1} dt \)). The resulting distribution \( Q \) satisfies the Hölder condition.
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