β matrix models in the multi cut regime

M.Shcherbina

Institute for Low Temperature Physics, Kharkov, Ukraine

Paris

Model definition

Distributions in $\mathbb{R}^n$, depending on the function V and $\beta > 0$

$$p_{n,\beta}(\lambda_1, ..., \lambda_n) = Z_{n,\beta}^{-1}[V] e^{\beta H(\lambda_1,...,\lambda_n)/2},$$

where H (Hamiltonian) and $Z_{n,\beta}[V]$ (partition function) are

$$H(\lambda_1, \ldots, \lambda_n) = -\, n \sum_{i=1}^{n} V(\lambda_i) + \sum_{i \neq j} \log|\lambda_i - \lambda_j|,$$

$$Z_{n,\beta}[V] = \int e^{\beta H(\lambda_1,...,\lambda_n)/2} d\lambda_1 \ldots d\lambda_n, \quad V(\lambda) > (1+\varepsilon)\log(1+\lambda^2).$$

For $\beta = 1, 2, 4$ it is a joint eigenvalues distribution of real symmetric, hermitian and symplectic matrix models respectively.

Marginal densities (correlation functions)

$$p_l^{(n)}(\lambda_1, ..., \lambda_l) = \int_{\mathbb{R}^{n-l}} p_{n,\beta}(\lambda_1, ...\lambda_l, \lambda_{l+1}, ..., \lambda_n) d\lambda_{l+1} ... d\lambda_n$$

Linear eigenvalue statistics and characteristic functional

Linear eigenvalue statistics (LES) of the test function h and NCM

$$\mathcal{N}_n[h] = \sum_{i=1}^{n} h(\lambda_i), \quad N_n[\Delta] = \sharp\{\lambda_i \in \Delta\}/n$$

The moments of LES can be written in terms of correlation functions. In particular,

$$E\{\mathcal{N}_n[h]\} = n \int h(\lambda) p_1^{(n)}(\lambda) d\lambda$$

and $\mathrm{Var}_n\{\mathcal{N}_n[h]\}$ can be expressed in terms of $p_2^{(n)}(\lambda_1, \lambda_2)$ and $p_1^{(n)}(\lambda_1)$.

Characteristic functional

$$\begin{aligned}\widetilde{Z}_{n,\beta}[h] =& E_{\beta,n}\left\{ e^{\beta(\mathcal{N}_n[h] - E\{\mathcal{N}_n[h]\})/2} \right\} \\ =& Z_{n,\beta}[V - \frac{1}{n}(h - E\{\mathcal{N}_n[h]\})]/Z_{n,\beta}[V],\end{aligned}$$

hence we study the partition function of the perturbed potential

First step for β matrix models

Theorem [Boutet de Monvel, Pastur, S:95; Johansson:98]

If V is a Hölder function, then

$$\log Z_{n,\beta}[V] = \frac{n^2\beta}{2}\mathcal{E}[V] + O(n\log n),$$

where $\mathcal{E}[V] = -\min_{m\in\mathcal{M}_1}\left\{L[dm, dm] + \int V(\lambda)m(d\lambda)\right\} = \mathcal{E}_V(m^*)$,

$$L[\,dm, dm'] = \int \log|\lambda-\mu|^{-1} dm(\lambda)dm'(\mu),$$

$m^*(d\lambda) = \rho(\lambda)d\lambda$ (called the equilibrium measure) has a compact support $\sigma := \operatorname{supp} m^*$.
Moreover, if $h' \in L_2[\sigma_\varepsilon]$

$$|n^{-1}E\{\mathcal{N}_n[h]\} - (h, m^*)| \le Cn^{-1/2}\log^{1/2} n||h'||_2^{1/2}||h||_2^{1/2}$$

Motivation to study $\log Z_{n,\beta}[V]$: universality for $\beta = 1, 4$

Result of Widom:99

For polynomial V of degree 2m there is $(2m-1) \times (2m-1)$ matrix T_n (it can be constructed directly) such that if $\log\det T_n > -C$ uniformly in n, then the Dyson universality conjecture is true for $\beta = 1, 4$

- $V = \lambda^4/4 + a\lambda^2/2$ [Stojanovich:02],
- $V = \lambda^{2m}$ [Deift,Gioev:07,07a],
- V-real analytic with one interval equilibrium density [S:09,09a].

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Observation of Stojanovich [St:02]

$$\det(T_n) = \left(\frac{Z_{n,1}[V]Z_{n/2,4}[V]}{Z_{n,2}[V](n/2)!2^n}\right)^2$$

Hence to control $\det(T_n)$, it suffices to control $\log(Z_{n,\beta}/n!)$ for $\beta = 1, 2, 4$ up to $O(1)$ terms.

Results for one cut potentials

Theorem [Johansson:98] CLT for LES in the one cut case

V is polynomial, $\sigma = [a, b]$, and ρ is "generic". Then for any $h : \mathbb{R} \to \mathbb{R}$ with $||h^{(4)}||_\infty, ||h'||_\infty \le \log n$

$$\widetilde{Z}_{n,\beta}[h] = \exp\Big\{\frac{\beta}{2}\Big(\big(\frac{2}{\beta} - 1\big)(h, \nu) + \frac{1}{4}(\overline{D}_\sigma h, h)\Big)\Big\}\Big(1 + n^{-1}O\big(||h^{(4)}||_\infty^3\big)\Big)$$

where the "variance operator" $\overline{D}_\sigma$ and the measure ν have the form

$$(\overline{D}_\sigma h, h) = \int_\sigma \frac{h(\lambda)d\lambda}{\pi^2 X^{1/2}(\lambda)} \int_\sigma \frac{h'(\mu)X^{1/2}(\mu)d\mu}{\lambda - \mu}, \quad X_\sigma(\lambda) = (b - \lambda)(\lambda - a)$$

$$(\nu, h) := \frac{1}{4}(h(b) + h(a)) - \frac{1}{2\pi}\int_\sigma \frac{h(\lambda)d\lambda}{X^{1/2}(\lambda)} + \frac{1}{2}(D_\sigma \log P, h)$$

Remark

D_σ is "almost" $\mathcal{L}_\sigma^{-1}$, where $\mathcal{L}_\sigma$ is the integral operator defined by the kernel $\log|\lambda - \mu|^{-1}$ for the interval σ

Theorem [Kriecherbauer, S:10]

1. For $h = 0$

$$\log(Z_{n,\beta}/n!) = \frac{\beta n^2}{2}\mathcal{E}[V] + F_\beta(n) + n\Big(\frac{\beta}{2}-1\Big)\big((\log\rho,\rho) - 1 - \log 2\pi\big) + r_\beta[\rho] + O(n^{-1}),$$

where $F_\beta(n)$ corresponds to the linear, logarithmic and zero order terms of the expansion in n of $\log Z_{n,\beta}[V^*]$ for $V^*(\lambda) = \lambda^2/2$:

$$F_\beta(n) = n\Big(\frac{\beta}{2}-1\Big)\Big(\log\frac{n\beta}{2} - \frac{1}{2}\Big) + n\log\frac{\sqrt{2\pi}}{\Gamma(\beta/2)} - c_\beta\log n + c_\beta^{(1)},$$

where $c_\beta = \frac{\beta}{24} - \frac{1}{4} + \frac{1}{6\beta}$ and $c_\beta^{(1)}$ is some depending only on β constant (for $\beta = 2$, $c_\beta^{(1)} = \zeta'(1)$)

Other results for one cut potentials:

1. [Albeverio,Pastur,S:01] expansion of the first and the second correlators for one-cut real analytic V and $\beta = 2$;

2. [Ercolani, McLaughlin:03] expansion of $\log Z_{n,\beta}[V]$ for polynomial one-cut V and $\beta = 2$;

3. [Borot, Guionnet:11] expansion of all correlators and $\log Z_{n,\beta}[V]$ for one-cut real analytic V and any β.

CLT and expansions for multi - cut case. Results.

1. [Chekhov,Eynard:06, Eynard:09] formal expansions for multi-cut V and any β;

2. [Pastur:07] derivation of CLT from OP-asymptotics of [Deift at al:99] for real analytic h and $\beta = 2$;

Idea from the mean field theory of statistical mechanics

Consider the Hamiltonian

$$H_n(\bar\sigma) = H_n^*(\bar\lambda) + \frac{1}{2}\Big(\sum_j \varphi(\lambda_j)\Big)^2,$$

where H_n^* is the Hamiltonian for which we are able to find the $\log Z_n^*(u)$ up to the order $O(n^{-k})$

$$f_n^*(u) = \beta^{-1}\log Z_n^*(u) = \beta^{-1}(\log Z_n^*(0) + d\frac{u^2}{2} + \sum d_k(u)n^{-k})$$

where

$$Z_n^*[u] := \int d\bar\lambda e^{\beta(H_n(\bar\lambda)+nv\sum\varphi(\lambda_i)+u\sum(\varphi(\lambda_i)-\langle\varphi\rangle))}.$$

We would like to find the partition function of H_n:

$$Z_n = \int d\bar\lambda e^{\beta H_n(\bar\lambda)}$$

- Introduce the "approximate" Hamiltonian

$$H_n^{(a)}(\bar\lambda) = H_n^*(\bar\lambda) + nv\sum_j \varphi(\lambda_i) - n^2\frac{v^2}{2}, \quad v = \langle\varphi\rangle_{H_n}$$

$$H_n = H_n^{(a)}(\bar\lambda) + \frac{1}{2}\sum_{i,j}(\varphi(\lambda_i) - \langle\varphi\rangle_{H_n})(\varphi(\lambda_j) - \langle\varphi\rangle_{H_n})$$

- Use the Hubbard-Stratonovich transformation

$$Z_n = \sqrt{\frac{\beta}{2\pi}}\int d\bar\lambda\int du e^{\beta H_n^{(a)}(\bar\lambda) + u\beta\sum_j(\varphi(\lambda_i) - \langle\varphi\rangle_{H_n}) - \beta u^2/2}$$

- Take the integral with respect to $\bar\lambda$ first. We obtain

$$Z_n = \int du e^{\beta f_n^*(u) - \beta u^2/2}$$

- Expand $f_n^*(u)$ in the series with respect to n^{-1} and live at the exponent only O(1) quadratic term. Taking the integrals with respect to u we obtain the expansion for Z_n.

Assumptions and the restriction of the integration domain

Assumptions

1. V is real analytic,

$$\sigma = \bigcup_{\alpha=1}^{q} \sigma_\alpha, \quad \mu_\alpha = \int_{\sigma_\alpha} \rho_\alpha(\lambda) d\lambda, \quad \rho_\alpha := 1_{\sigma_\alpha} \rho.$$

2. V is of generic behavior.

Replace the integration domain in the definition of $Z_{n,\beta}[V]$ from $\mathbb{R}$ to σ_ε, where

$$\sigma_\varepsilon = \bigcup_{\alpha=1}^{q} \sigma_{\alpha,\varepsilon}, \quad \sigma_{\alpha,\varepsilon} \cap \sigma_{\alpha+1,\varepsilon} = \emptyset$$

Then, according to the result of [Pastur,S:07], $Z_{n,\beta}[V]$ will be changed by $(1 + O(e^{-nc}))$ factor.

Construction of the "approximate" Hamiltonian

Below we will use the notation

$$\chi_\alpha(\lambda) = 1_{\sigma_{\alpha,\varepsilon}}(\lambda),$$

Then for our domain $\quad 1 = \sum_\alpha \chi_\alpha(\lambda)$ and we can write $H(\lambda)$ as

$$\begin{aligned} H(\overline{\lambda}) &= -\,n\sum_{i=1}^{n} V(\lambda_i) + \sum_{i\neq j,\alpha,\alpha'=1}^{q} \chi_\alpha(\lambda_i)\chi_{\alpha'}(\lambda_j)\log|\lambda_i-\lambda_j| \\ &= -\,n\sum_{i=1}^{n} V(\lambda_i) + \sum_{i\neq j}\sum_{\alpha=1}^{q} \chi_\alpha(\lambda_i)\chi_\alpha(\lambda_j)\log|\lambda_i-\lambda_j| \\ &\quad + \sum_{\substack{i,j=1\\ \alpha\neq\alpha'}}^{n} \int \log|\lambda-\mu|\chi_\alpha(\lambda)\chi_{\alpha'}(\mu)\delta_{\lambda_i}(\lambda)\delta_{\lambda_j}(\mu)d\lambda d\mu = H^* \end{aligned}$$

Then write under the integral sign

$$\delta_{\lambda_i}(\lambda) = \delta_{\lambda_i}(\lambda) - \langle\delta_{\lambda_i}(\lambda)\rangle + \langle\delta_{\lambda_i}(\lambda)\rangle = \Delta_i(\lambda) + \langle\delta_{\lambda_i}(\lambda)\rangle$$

Construction of the "approximate" Hamiltonian

$$\begin{aligned} H(\bar\lambda) =& H^*(\bar\lambda) + 2n\sum_{j=1}^{n} V_\alpha^{(a)}(\lambda_i) - n^2\Sigma^* \\ &+ \sum_{\substack{i,j=1\\ \alpha\neq\alpha'}}^{n}\int \log|\lambda-\mu|\chi_\alpha(\lambda)\chi_{\alpha'}(\mu)\Delta_i(\lambda)\Delta_j(\mu)d\lambda d\mu \\ =& H_a(\bar\lambda) + \Delta H(\bar\lambda), \end{aligned}$$

where, taking into account that $\langle\delta_{\lambda_i}(\lambda)\rangle = \rho(\lambda)$, we obtain that the "effective potentials" $V_\alpha^{(a)}$ and Σ^* have the form

$$\begin{aligned} V_\alpha^{(a)}(\lambda) =& \chi_\alpha(\lambda)\sum_{\alpha'\neq\alpha}\int \log|\lambda-\mu|\chi_{\alpha'}(\mu)\rho(\mu)d\mu \\ \Sigma^* :=& \sum_{\alpha\neq\alpha'}\int_{\sigma_\alpha} d\lambda\int_{\sigma_{\alpha'}} d\mu \log|\lambda-\mu|\rho(\lambda)\rho(\mu) \end{aligned}$$

It is easy to see that

$$Z_{n,\beta}[V]/n! = \sum_{|\bar{n}|=n} \frac{\int 1_{\overline{n}}(\bar{\lambda})e^{\beta H(\bar{\lambda})/2}}{n_1!\dots n_q!} = \sum_{|\bar{n}|=n} \frac{\int 1_{\overline{n}}(\bar{\lambda})e^{\beta(H_a(\bar{\lambda})+\Delta H(\bar{\lambda}))/2}}{n_1!\dots n_q!}$$

where $\overline{n} := (n_1, \dots, n_q)$, $|\bar{n}| := \sum_{\alpha=1}^{q} n_\alpha$, and $1_{\overline{n}}(\bar{\lambda})$ is the indicator of the configurations of $\lambda_1, \dots, \lambda_n$, such that
$\lambda_1, \dots, \lambda_{n_1} \in \sigma_{1,\varepsilon}$, $\lambda_{n_1+1}, \dots, \lambda_{n_1+n_2} \in \sigma_{2,\varepsilon}$, $\dots$ $\lambda_{n-n_q+1}, \dots, \lambda_q \in \sigma_{q,\varepsilon}$
Choose $M = [\log^2 n]$ and represent $\Delta H(\bar{\lambda})1_{\overline{n}}(\bar{\lambda})$ as

$$1_{\overline{n}}(\bar{\lambda}) \sum_{\substack{j,j'=1\\ \alpha\neq\alpha'}}^{n} \sum_{k,m=1}^{M} L_{k,m}^{(\alpha,\alpha')}\Big(p_k^{(\alpha)}(\lambda_j) - \frac{n}{n_\alpha}c_k^{(\alpha)}\Big)\Big(p_m^{(\alpha')}(\lambda_{j'}) - \frac{n}{n_{\alpha'}}c_k^{(\alpha)}\Big)$$

$$+ O(e^{-c\log^2 n}), \quad c_k^{(\alpha)} := (p_k^{(\alpha)}, \rho 1_{\sigma_\alpha})$$

where $L_{k,m}^{(\alpha,\alpha')}$ are the Fourier coefficient of the function
$\log|\lambda - \mu|\chi_\alpha(\lambda)\chi_{\alpha'}(\mu)$ with respect to the basis $\{p_k^{(\alpha)}(\lambda)p_m^{(\alpha')}(\mu)\}$

Main steps of the proof

- For each $\bar{n}$ we linearize $\Delta H(\bar{\lambda})1_{\bar{n}}(\bar{\lambda})$, using the Hubbard-Stratonovich transformation. This adds to $H^{(a)}(\bar{\lambda})1_{\bar{n}}(\bar{\lambda})$ the additional potential $h_\alpha[\bar{u}]$, depending on the integration parameters $\bar{u}$. Then for each σ_α we are in the situation of Theorem 3.
- Apply Theorems 2,3 to find $Z_{n_\alpha,\beta}[V + 2V_\alpha^{(a)} + \frac{1}{n}h_\alpha[\bar{u}]]$. We obtain the quadratic form of $\bar{u}$ in the exponent.
- Integrate with respect to $\bar{u}$. We obtain the expansion for the initial partition function.

Important definitions

$$X_\sigma(\lambda) = \prod_{\alpha=1}^{q} (b_\alpha - \lambda)(\lambda - a_\alpha)$$

Definition of $\mathcal{Q}$

$$\mathcal{Q} = \{\mathcal{Q}_{\alpha\alpha'}\}_{\alpha,\alpha'=1}^{q}, \quad \mathcal{Q}_{\alpha\alpha'} = (\mathcal{L}\psi^{(\alpha)}, \psi^{(\alpha')}),$$

where $\psi^{(\alpha)}(\lambda) = p_\alpha(\lambda)X^{-1/2}(\lambda)1_\sigma$ (p_α is a polynomial of degree $q-1$) is a unique solution of the system of equations

$$(\mathcal{L}\psi^{(\alpha)})_{\alpha'} = \delta_{\alpha\alpha'}, \quad \alpha' = 1, \ldots, q.$$

(harmonic measure of σ_α with respect to $\mathbb{C} \setminus \sigma$)

Definition of I[h]

$$I[h] = (I_1[h], \ldots, I_q[h]), \quad I_\alpha[h] := \sum_{\alpha'} \mathcal{Q}^{-1}_{\alpha\alpha'}(h, \psi^{(\alpha')}).$$

Main results

Theorem 3 [S:12]

Let the potential V satisfy conditions C1-C2. Then

1. for $h : ||h^{(6)}||_\infty < \infty$

$$\widetilde{Z}_{n,\beta}[h] = e^{\frac{\beta}{8}(\mathcal{D}h,h)+\left(\frac{\beta}{2}-1\right)(\mathcal{G}\nu,h)}\frac{\Theta(\bar{I}[h];\{n\bar{\mu}\})}{\Theta(0;\{n\bar{\mu}\})} \times \left(1+O\left(n^{-\kappa}(||h'||_\infty ||h^{(6)}||_\infty^2)\right)\right),$$

where the operators $\mathcal{D}$, $\mathcal{G}$, $\widetilde{\mathcal{L}}$ are defined in terms of $\mathcal{L}$ and $\oplus D_{\sigma_\alpha}$

$$\Theta(I[h];\{n\bar{\mu}\}) := \sum_{n_1+\cdots+n_q=n_0} \exp\Big\{-\frac{\beta}{2}\Big(\mathcal{Q}^{-1}\Delta\bar{n},\Delta\bar{n}\Big)+\frac{\beta}{2}(\Delta\bar{n},I[h]) + (\frac{\beta}{2}-1)(\Delta\bar{n},I[\log\overline{\rho}])\Big\},$$

$$\{n\bar{\mu}\}=(\{n\mu_1\},\ldots,\{n\mu_q\}),\ (\Delta\bar{n})_\alpha = n_\alpha - \{n\mu_\alpha\},\ n_0=\sum^{q}\{n\mu_\alpha\},$$

1. for $h = 0$ we have

$$Z_{n,\beta}[V] = \mathcal{S}_{n,\beta}[V] \frac{\exp\left\{\frac{2}{\beta}\left(\frac{\beta}{2}-1\right)^2(\widetilde{\mathcal{L}}\mathcal{G}\nu,\nu)\right\}}{\det^{1/2}(1-\overline{D}\widetilde{\mathcal{L}})}\Theta(0;\{n\bar{\mu}\})(1+O(n^{-\kappa})),$$

$$\mathcal{S}_{n,\beta}[V] = \exp\left\{\frac{n^2\beta}{2}\mathcal{E}[V] + F_\beta(n) + n(\frac{\beta}{2}-1)\big((\log\rho,\rho) - 1 - \log 2\pi\big) - c_\beta(q-1)\log n + \sum_{\alpha=1}^{q}(r_\beta[\mu_\alpha^{-1}\rho_\alpha] - c_\beta\log\mu_\alpha)\right\},$$

Theorem 4 [S:in prep]

Under the conditions C1-C2 $Q_{n,\beta}[V]$ admits the asymptotic expansion in n^{-j} with quasi-periodic in n coefficients $q_{\beta,j}[n]$:

$$Z_{n,\beta}[V] = \mathcal{S}_{n,\beta}[V] \frac{\exp\left\{\frac{2}{\beta}\left(\frac{\beta}{2}-1\right)^2(\widetilde{\mathcal{L}}\mathcal{G}\nu,\nu)\right\}}{\det^{1/2}(1-\overline{D}\widetilde{\mathcal{L}})}\Theta(0;\{n\bar{\mu}\})\sum_{j=1} n^{-j}q_{\beta,j}[n],$$

Corollaries

Corollary 1

Theorem 3 yields that the fluctuations of $\mathcal{N}_n[h]$ for generic h are non Gaussian. They are Gaussian, if there exists some c such that

$$I_\alpha[h] = c, \quad \alpha = 1, \dots, q; \quad \Leftrightarrow \quad (h - c, \psi^{(\alpha)}) = 0, \quad \alpha = 1, \dots, q.$$

Remark 1

The operator $\mathcal{D}$, which appears in the place of the "variance operator" in the multi-cut case, differs from $\mathcal{L}^{-1}$ only by the finite rank perturbation. This perturbation provides, in particular, that $\mathcal{D}f = 0$, if $f(\lambda) = \text{const}$, $\lambda \in \sigma$

$$(\mathcal{D}h, h) = \frac{1}{\pi^2} \int_\sigma \frac{h(\lambda) d\lambda}{X^{-1/2}(\lambda)} \int_\sigma \frac{h'(\mu) X^{1/2}(\mu) d\mu}{(\lambda - \mu)} - ?$$

Corollaries of Theorem 3 for the moments of $\mathcal{N}_n[h]$

Expectation of $\mathcal{N}_n[h]$

For any β in the multi cut case we obtain $O(n^{-1})$ correction to $E_{\beta,n}\{n^{-1}\mathcal{N}_n[h]\}$:

$$E_{\beta,n}\{n^{-1}\mathcal{N}_n[h]\}-(h,\rho)=\frac{1}{n}\Big[\Big(\frac{\beta}{2}-1\Big)(\mathcal{G}\nu,h)+\sum_{\alpha=1}^{q}I_\alpha[h]c_\alpha(n)\Big]+O\big(\frac{1}{n^{1+\delta}}\big),$$

Variance of $\mathcal{N}_n[h]$

$$\mathrm{Var}\{\mathcal{N}_n[\varphi]\}=\frac{\beta}{8}(\mathcal{D}h,h)+\sum_{\alpha=1}^{q}d_{\alpha,\alpha'}(n)I_\alpha[h]I_{\alpha'}[h]$$

where $c_\alpha(n)$ and $d_{\alpha,\alpha'}(n)$ are quasi-periodic functions (derivatives of Θ-function above).

Corollaries from Theorem 3 for the universality of local regimes

1. bulk universality for $\beta = 1, 4$;
2. edge universality for $\beta = 1, 4$;
3. bulk universality for any β could be reduced to the universality for the one-cut case for $\mathrm{V} + \mathrm{n}^{-1}\mathrm{h}$ with $||\mathrm{h}'||_\infty \leq \log \mathrm{n}$.