Hausdorff dimension for subshifts invariant under the multiplicative integers

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Multiplicative golden mean shift

Consider

$$\mathcal{M}_G := \left\{ x = \sum_{k=1}^{\infty} x_k 2^{-k} : \ x_k \in \{0,1\}, \ x_k x_{2k} = 0 \text{ for all } k \right\}$$

[A. Fan, L. Liao, J. Ma] computed the Minkowski dimension and raised the question of computing the Hausdorff dimension of $\mathcal{M}_G$.

- Invariance under $\mathbb{N}^*$ - the multiplicative semigroup:

$$(x_k) \mapsto (x_{rk}) \ \text{ for } r \in \mathbb{N}.$$

- Motivation from “multifractal analysis of double averages” (will discuss at the end)

$\mathcal{M}_G = \{(x_k)_1^\infty \in \{0,1\}^{\mathbb{N}} : \; x_k x_{2k} = 0, \; k \ge 1\}$

Theorem

(i) [Fan, Liao, Ma]

$$\dim_M(\mathcal{M}_G) = \sum_{k=1}^{\infty} 2^{-k-1} \log_2 F_{k+1} = 0.82429\ldots,$$

where F_k is the k-th Fibonacci number: $F_1 = 1, \; F_2 = 2, \ldots$

(ii) [KPS]

$$\dim_H(\mathcal{M}_G) = -\log_2 p = 0.81137\ldots,$$

where $p^3 = (1-p)^2, \; 0 < p < 1$.

Compare with the usual, “additive” golden mean shift:

$$G := \Big\{x = \sum_{k=1}^{\infty} x_k 2^{-k} :\ x_k \in \{0,1\},\ x_k x_{k+1} = 0 \text{ for all } k\Big\},$$

$$\dim_H(G) = \dim_M(G) = \log_2\Big(\frac{1+\sqrt{5}}{2}\Big)$$

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More generally, [Furstenberg, 1967]: if X is a closed subset of $[0,1]$, invariant under $T_m :\ x \mapsto mx \pmod 1$, then

$$\dim_H(X) = \dim_M(X) = \frac{h_{\mathrm{top}}(T_m|_X)}{\log m}$$

General multiplicative SFT

Theorem

Let $A = (A(i,j))_{i,j=0}^{m-1}$ *be a primitive 0-1 matrix,*

$$\mathcal{M}_A := \left\{(x_k)_1^\infty \in \{0,\ldots,m-1\}^{\mathbb{N}},\ A(x_k, x_{2k}) = 1, k \geq 1\right\}.$$

(i)

$$\dim_H(\mathcal{M}_A) = \frac{1}{2}\log_m \sum_{i=0}^{m-1} t_i,$$

where

$$t_i^2 = \sum_{j=0}^{m-1} A(i,j)t_j,\ \ t_i > 1,\ \ i \leq m.$$

Theorem (cont.)

(ii) The Minkowski dimension of $\mathcal{M}_A$ *exists and equals*

$$\dim_M(\mathcal{M}_A) = \sum_{k=1}^{\infty} \frac{\log_m(A^{k-1}\overline{1}, \overline{1})}{2^{k+1}}$$

where $\overline{1} = (1, \ldots, 1)^T \in \mathbb{R}^m$*.*

(iii) $\dim_H(\mathcal{M}_A) = \dim_M(\mathcal{M}_A)$ *if and only if all row sums of* A *are equal.*

Analogy with self-affine sets

Bedford (1984) and McMullen (1984) computed the dimension of self-affine carpets associated to a 0-1 rectangular $m \times n$ matrix, with $n > m$.

- In most cases, the Hausdorff dimension is less than the Minkowski dimension.
- The dimensions are equal iff all row sums of the matrix are equal (uniform horizontal fibers).
- There are many parallels (including methods used in the proof) with the multiplicative subshifts, but no direct logical connection, as far as we know.

Proof sketch: Minkowski dimension of $\mathcal{M}_G$

Need to estimate $N_n(\mathcal{M}_G)$, the number of words of length n. Assume WLOG that $n = 2^\ell d$ for large ℓ.

- $x_i \in \{0, 1\}$ for odd $i \in (n/2, n]$

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- $x_i \ldots x_{2^{k-1}i} \in \{$ words of length k in $G\}$ for odd $i \in (\frac{n}{2^k}, \frac{n}{2^{k-1}}]$, and there are $\frac{n}{2^{k+1}}$ such odds, so

$$N_n(\mathcal{M}_G) = 2^{n/4} \cdot 3^{n/8} \cdot \ldots \cdot F_k^{n/2^{k+1}} \ldots,$$

$$\dim_M(\mathcal{M}_G) = \lim_{n\to\infty} \frac{\log_2 N_n(\mathcal{M}_G)}{n} = \sum_{k=1}^{\infty} \frac{\log_2 F_k}{2^{k+1}}$$

□

How to compute the Hausdorff dimension?

$\Sigma_2 = \{0,1\}^{\mathbb{N}}$ with $\varrho(x,y) = 2^{-\min\{n:\, x_n \ne y_n\}}$, $x_1^n = x_1 \dots x_n$.

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Lemma (Billingsley)

Let ν be a finite Borel measure on a Borel $E \subset \Sigma_2$.
(i) If

$$\liminf_{n\to\infty} \frac{-\log_2 \nu[x_1^n]}{n} \ge s \text{ for } \nu\text{-a.e. } x \in E,$$

then $\dim_H(E) \ge s$.

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$$\liminf_{n\to\infty} \frac{-\log_2 \nu[x_1^n]}{n} \le s \text{ for } \textbf{all } x \in E,$$

then $\dim_H(E) \le s$.

How to put a measure on $\mathcal{M}_G$?

CRUCIAL OBSERVATION:

$$\mathcal{M}_G = \{x \in \Sigma_2 : \; x|_{J(i)} \in G\} \;\text{ for all odd } i,$$

where $J(i) = \{i, 2i, 4i, \ldots\} = \{2^r i\}_{r=0}^{\infty}$.

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Take **any** probability measure μ on G and then let

$$\mathbb{P}_\mu[u] := \prod_{i \le n,\, i \text{ odd}} \mu[u|_{J(i)}], \quad |u| = n.$$

Then $\mathbb{P}_\mu$ is a probability measure on $\mathcal{M}_G$.

Proof sketch of the lower bound

We have

$$\mathbb{P}_\mu[x_1^n] \le \prod_{n/2<i\le n,\ i \text{ odd}} \mu[x_i] \prod_{n/4<i\le n/2,\ i \text{ odd}} \mu[x_i x_{2i}] \cdots$$

That is (assuming n is divisible by 2^ℓ),

$$\mathbb{P}_\mu[x_1^n] \le \prod_{k=1}^{\ell} \prod_{\frac{n}{2^k}<i\le\frac{n}{2^{k-1}},\ i \text{ odd}} \mu[x_1^n|_{J(i)}].$$

For $\frac{n}{2^k} < i \leq \frac{n}{2^{k-1}}$, the words $x_1^n|_{J(i)}$ are i.i.d., and there are $\frac{n}{2^{k+1}}$ of them.

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By the SLLN, almost surely,

$$\sum_{\frac{n}{2^k}<i\le\frac{n}{2^{k-1}},\ i\ \text{odd}} \frac{-\log_2 \mu[x_1^n|_{J(i)}]}{(n/2^{k+1})} \to H^\mu(\alpha_k) \ \text{ as } n = 2^\ell r \to \infty,$$

where α_k is the partition of G into cylinders of length k and

$$H^\mu(\alpha) = -\sum_{A\in\alpha} \mu(A) \log_2 \mu(A).$$

Thus

$$\liminf_{n\to\infty} \frac{-\log_2 \mathbb{P}_\mu[x_1^n]}{n} \geq \sum_{k=1}^{\ell} \frac{H^\mu(\alpha_k)}{2^{k+1}},$$

hence Billingsley's Lemma implies

Proposition

$$\dim_H(\mathcal{M}_G) \geq s(\mu) := \sum_{k=1}^{\infty} \frac{H^\mu(\alpha_k)}{2^{k+1}}.$$

"Optimal" measure for $\mathcal{M}_G$

Clearly, $\dim_H(\mathcal{M}_G) \geq \max_\mu s(\mu)$, so need to find the maximizing, "optimal" measure.

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Conditioning on the first digit, by elementary properties of entropy, we obtain that $s = \max\{s(\mu) : \ \mu(G) = 1\}$ satisfies

$$s = \frac{H(p)}{2} + \frac{ps}{2} + \frac{(1-p)s}{4},$$

where $p = \mu[0]$.

Proof of the lower bound for $\dim_H(\mathcal{M}_G)$

It follows that

$$s = \frac{2H(p)}{3-p},$$

and maximizing over p yields $p^3 = (1-p)^2,\ s = -\log_2 p$. Thus $\dim_H(\mathcal{M}_G) \geq s$. □

We obtain that the optimal measure μ on G is Markov (not stationary!), with

$$\mathbf{p}_{\text{init}} = (p, 1-p), \quad P_{\text{trans}} = \begin{pmatrix} p & 1-p \\ 1 & 0 \end{pmatrix}.$$

Proof sketch of the upper bound

Recall that the optimal μ on G is Markov, with

$$\mathbf{p}_{\mathrm{init}} = (p, 1-p), \quad P_{\mathrm{trans}} = \begin{pmatrix} p & 1-p \\ 1 & 0 \end{pmatrix}.$$

Thus

$$\mu[u] = (1-p)^{N_1(u_1 \ldots u_k)} p^{N_0(u_1 \ldots u_k) - N_1(u_1 \ldots u_{k-1})},$$

where $u = u_1 \ldots u_n$ and $N_i(u)$ is the number of i's in u.

Proof sketch of the upper bound (cont.)

By the definition of $\mathbb{P}_\mu$, for n even, **for all** $x \in \mathcal{M}_G$,

$$\mathbb{P}_\mu[x_1^n] = (1-p)^{N_1(x_1^n)} p^{N_0(x_1^n) - N_1(x_1^{n/2})}$$

Using that $(1-p)^2 = p^3$ and $N_0(x_1^n) = n - N_1(x_1^n)$,

$$\mathbb{P}_\mu[x_1^n] = p^n p^{N_1(x_1^n)/2 - N_1(x_1^{n/2})},$$

$$-\frac{1}{n}\log_2 \mathbb{P}_\mu[x_1^n] = -\log_2 p \left(1 + \frac{1}{2}\left[\frac{N_1(x_1^n)}{n} - \frac{N_1(x_1^{n/2})}{n/2}\right]\right).$$

Proof sketch of the upper bound (end)

Let $n = 2^\ell$,

$$a_\ell = -\frac{1}{n}\log_2 \mathbb{P}_\mu[x_1^n].$$

The average of a_ℓ's "telescopes"!

$$\frac{a_1 + \cdots + a_\ell}{\ell} \to -\log_2 p,$$

hence

$$\liminf_{\ell\to\infty} a_\ell \le -\log_2 p = s,$$

and therefore, $\dim_H(\mathcal{M}_G) \le s$ by Billingsley's Lemma. $\square$

Multifractal analysis of Birkhoff averages

For a dynamical system (X, T), study

$$\theta \mapsto \dim_H \left\{ x \in X : \lim_{n\to\infty} \frac{S_n f(x)}{n} = \theta \right\}.$$

where $S_n f(x) = \sum_{k=1}^{n} f(T^k x)$.

Investigated by **Barral, Barreira, Fan, Feng, Liao, Mensi, Olsen, Peyrière, Pesin, Saussol, Seuret, Schmeling**, and others...

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Simplest example [Besicovitch 1934], [Eggleston, 1949]:

$$\dim_H \Big\{ (x_k)_1^\infty \in \Sigma_2 : \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} x_k = \theta \Big\} = H(\theta),$$

Multiple ergodic averages

For a dynamical system (X, T) consider

$$\frac{1}{n} S_n(f_1, \ldots, f_\ell)(x) := \frac{1}{n} \sum_{k=1}^{n} f_1(T^k x) f_2(T^{2k} x) \cdots f_\ell(T^{\ell k} x)$$

Their limits were studied by Furstenberg (1977), Bourgain (1990), Host and Kra (2005), and others.

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Seems very complicated, so consider the simplest cases...

Multiple ergodic averages (cont.)

[Fan, Liao, Ma] studied the case $X = \{-1, 1\}^{\mathbb{N}}$, T is the shift, $f_1(x) \equiv \ldots \equiv f_\ell(x) = x_1$, namely,

$$B_\theta := \left\{ (x_k)_1^\infty \in \{-1, 1\}^{\mathbb{N}} : \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} x_k x_{2k} \cdots x_{\ell k} = \theta \right\}$$

They proved that

$$\dim_H(B_\theta) = 1 - \frac{1}{\ell} + \frac{1}{\ell} H\left(\frac{1+\theta}{2}\right)$$

using Riesz products and the group properties of $\{-1, 1\}^{\mathbb{N}}$.

Multifractal analysis of double averages (cont.)

[Fan, Liao, Ma] raised the question about

$$A_\theta := \left\{ (x_k)_1^\infty \in \{0,1\}^{\mathbb{N}} : \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} x_k x_{2k} = \theta \right\}.$$

What is $\dim_H(A_\theta)$? Easy to see that $\dim_H(A_0) = \dim_H(\mathcal{M}_G)$

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Similar techniques yield:

$$\dim_H(A_\theta) = -\log_2[p^{1-\theta}(1-p)^{\theta/2}(1-q)^{\theta/2}],$$

where $(1-p)^2 q = p^3$, $\theta = \frac{2(1-p)(1-q)}{2-p+q}$.

Open Question

Compute the dimensions of

$$\widetilde{\mathcal{M}} := \left\{(x_k)_1^\infty \in \Sigma_2 : \; x_k x_{2k} x_{3k} = 0, k \geq 1\right\}$$

General set-up

Let $\Omega \subset \Sigma_m = \{0, \ldots, m-1\}^{\mathbb{N}}$ be any closed subset (need not be shift-invariant). Multiplicative subshift:

$$\mathcal{M}_\Omega := \left\{ (x_k)_{k=1}^\infty \in \Sigma_m : \ (x_{i2^r})_{r=0}^\infty \in \Omega \ \text{ for all } i \text{ odd} \right\}$$

Given a probability μ on Ω, we define a probability on $\mathcal{M}_\Omega$ by

$$\mathbb{P}_\mu[u] := \prod_{i \le n,\, i \text{ odd}} \mu[u|_{J(i)}],$$

where $J(i) = \{2^r i\}_{r=0}^\infty$ and $[u] = [u_1 \ldots u_n]$ is a cylinder set.

Variational Principle: classical

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4. $\dim_H(\nu) = h(\nu)/\log m$
5. Variational Principle:

   $$\dim_H(\Upsilon) = \max\{\dim_H(\nu) : \ \nu \text{ is ergodic on } \Upsilon\}$$

Variational Principle: multiplicative

1. Multiplicative subshift $\mathcal{M}_\Omega \subset \Sigma_m$
2. Measure $\mathbb{P}_\mu$ on $\mathcal{M}_\Omega$

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2. Measure $\mathbb{P}_\mu$ on $\mathcal{M}_\Omega$
3. Pointwise dimension of $\mathbb{P}_\mu$
4. Dimension of $\mathbb{P}_\mu$
5. Variational Principle:

   $$\dim_H(\mathcal{M}_\Omega) = \max\{\dim_H(\mathbb{P}_\mu) : \ \mu \text{ is a probability on } \Omega\}$$

General result

Consider the **set of prefixes** $\mathrm{Pref}(\Omega)$ for the closed set Ω and solve the system of equations

$$t_u^2 = \sum_{j:\ uj\in \mathrm{Pref}(\Omega)}, \quad t_u > 1, \quad u \in \mathrm{Pref}(\Omega).$$

Then

$$\dim_H(\mathcal{M}_\Omega) = \log_m t_\emptyset,$$

where $\emptyset$ is the empty word. We have

$$\dim_H(\mathcal{M}_\Omega) < \dim_M(\mathcal{M}_\Omega)$$

iff every prefix of length k, for any given k, has the same number of continuations.