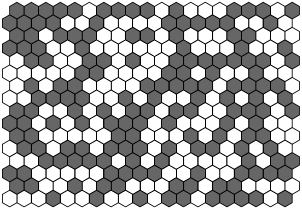
# Critical Percolation and Fractals

Jeff Steif

Fractals and Related Fields II
13 June 2011

## Percolation on the hexagonal lattice

Let each hexagon be black with probability $p$.



**Theorem:** (Harris, 1960)

If $p \leq 1/2$, then no infinite black component.

**Theorem:** (Kesten, 1980)

If $p > 1/2$, then there is an infinite black component.

The **critical value** is $1/2$.

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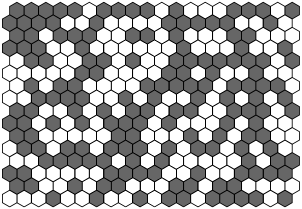

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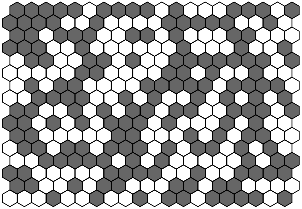

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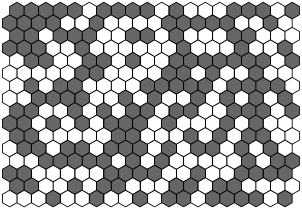

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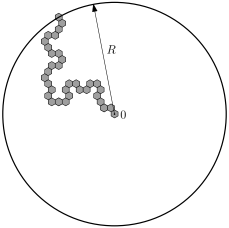
## The 1-arm exponent




Let $A_R$ be the event that there is an open path from the origin to **distance $R$ away**.

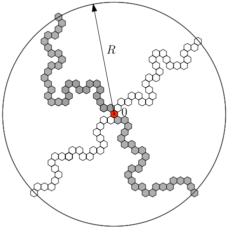
**Theorem. (Lawler, Schramm and Werner, 2002):**

For the hexagonal lattice, one has

$$P(A_R) = R^{-5/48+o(1)}$$

as $R \to \infty$.

## The 4-arm exponent




Let $A_R^4$ be the event that there are 4 paths adjacent to the origin to **distance $R$ away** alternating in color.

**Theorem. (Smirnov and Werner, 2001):**

For the hexagonal lattice, one has

$$P(A_R^4) = R^{-5/4+o(1)}$$

as $R \to \infty$.

## Tools for computing critical exponents

The above are two of an infinite number of **critical exponents** that one can compute.

These critical exponents were predicted by theoretical physicists.

**Conformal invariance** (Smirnov) and **Schramm-Lowner Evolution** (Schramm) are the tools used to derive these critical exponents.

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## Four fractals

We will look at 4 fractals associated to the above critical percolation.

1. The set $\mathcal{P}$ of **pivotal** hexagons for percolation crossing events.
2. **The spectral sample** $\mathcal{S}$ for percolation crossing events: Key to the study of **noise sensitivity**. Very related to (but different from!!) the set of **pivotal** hexagons.
3. **Fractal percolation:** a simpler model to keep in mind.
4. The set of exceptional times for **dynamical percolation**.

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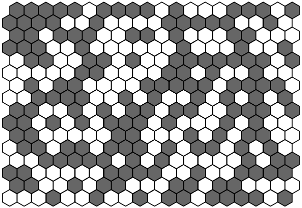
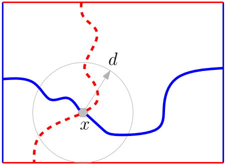
## The percolation crossing event

Through most of the talk, we are interested in the event that there is a **Left-Right crossing** of black hexagons in an $n \times n$ box.



## Pivotal hexagons and the pivotal set $\mathcal{P}$

A hexagon is **pivotal** if changing its status changes whether there is a L-R crossing: these are the hexagons which are important on a global scale.

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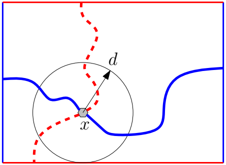


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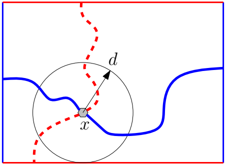


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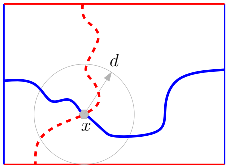


Note that this our **four arm event** and so the influence of each hexagon is about $\sim n^{-5/4}$. Hence $E(|\mathcal{P}|) \sim n^{3/4}$.

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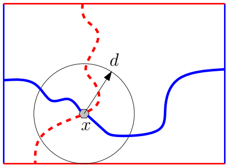


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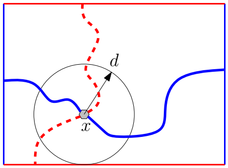


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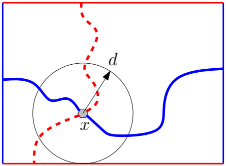


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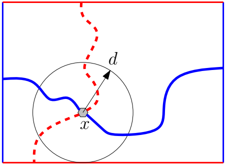


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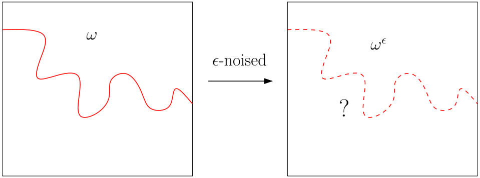
## Noise sensitivity for percolation

**Noise sensitivity** was introduced by **Benjamini, Kalai and Schramm**.



## Noise sensitivity for percolation: the question

Perform critical percolation on an $n \times n$ box in the hexagonal lattice.

Let $E_n$ be the event that there is a **L-R crossing** of black hexagons.

Fix $\epsilon > 0$ and **flip/reverse** the status of each hexagon with probability $\epsilon$.

Let $E_n^\epsilon$ be the event that there is a **L-R crossing** of black hexagons **after** the flipping procedure. (Of course $P(E_n^\epsilon) = P(E_n)$).

**Question:** Are $E_n$ and $E_n^\epsilon$ highly correlated or very independent?

(Of course, if $n$ is fixed and $\epsilon$ is small, they are highly correlated;

so we should think $\epsilon$ is **small** and **fixed** and $n \to \infty$.)

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## Noise sensitivity for percolation: some answers

**Theorem. (Benjamini, Kalai and Schramm, 1999):**

For all $\epsilon > 0$,

$$\lim_{n\to\infty} P(E_n \cap E_n^{\epsilon}) - P(E_n)^2 = 0.$$

**More quantitative versions**

**Theorem. (Schramm and S., 2010):**

If $\epsilon_n = (1/n)^{\gamma}$ with $\gamma < 1/8$, then

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**Theorem. (Garban, Pete and Schramm, 2010):**

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## Noise sensitivity for percolation: some remarks

- A crucial ingredient in **all** three proofs is **Fourier analysis**.
- Benjamini, Kalai and Schramm: exploit hypercontractivity.
- Schramm and S.: develop results in randomized algorithms.
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## The 3/4 exponent in a nutshell

- Recall that the probability that a hexagon is pivotal is $\sim n^{-5/4}$ and hence the expected number of pivotal hexagons is $\sim n^{3/4}$.
- Therefore $\epsilon_n = (1/n)^{3/4}$ should be the crossover point when we are likely to reverse a pivotal hexagon which "should" completely mix things up.

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## The Fourier set-up

The set of all functions $f : \{-1, 1\}^n \to R$ is a $2^n$ dimensional vector space with orthogonal basis $\{\chi_S\}_{S \subseteq \{1,\dots,n\}}$ where

$$\chi_S(x_1, \dots, x_n) := \prod_{i \in S} x_i.$$

We then can write

$$f = \sum_{S \subseteq \{1,\dots,n\}} \hat{\mathbf{f}}(\mathbf{S}) \chi_S \text{ with } \hat{\mathbf{f}}(\mathbf{S}) = E(f \chi_S).$$

If $f$ maps to $\{\pm 1\}$, then $E(f^2) = 1$ and

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## The Fourier set-up for percolation

Consider percolation on an $n \times n$ box and let $f_n$ be 1 if there is a L-R black crossing and -1 otherwise.

$f_n$ is a function of the states of the hexagons; i.e., (if we identify "black" with 1 and "white" with -1)

$$f_n : \{-1, 1\}^{H_n} \rightarrow \{\pm 1\}$$

where $H_n$ is the set of hexagons in the $n \times n$ box.

$$f_n = \sum_{S \subseteq H_n} \hat{\mathbf{f}}_{\mathbf{n}}(\mathbf{S}) \chi_S \text{ with } \hat{\mathbf{f}}_{\mathbf{n}}(\mathbf{S}) = E(f_n \chi_S).$$

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$$f_n = \sum_{S \subseteq H_n} \hat{\mathbf{f}}_{\mathbf{n}}(\mathbf{S}) \chi_S \text{ with } \hat{\mathbf{f}}_{\mathbf{n}}(\mathbf{S}) = E(f_n \chi_S).$$

**Definition:** The **spectral sample** for $f_n$, denoted by $\mathcal{S}_n$, is a random subset of $H_n$ whose distribution is given by

$$P(\mathcal{S}_{\mathbf{n}} = S) = \hat{f}_n^{\,2}(S).$$

So $\mathcal{S}_n$ is a random subset of $H_n$ where the subset $S$ is chosen with probability $\hat{f}_n^{\,2}(S)$.

$\mathcal{S}_n$ is our second fractal set.

## The Fourier picture and noise sensitivity

The **key connection** between the Fourier coefficients and noise sensitivity is the following elementary fact.

$$E(f_n(\omega)f_n(\omega^\epsilon)) = \sum_{S\subseteq H_n} \hat{f}_n(S)^2(1-2\epsilon)^{|S|}$$

$$= \mathbf{E}[(\mathbf{1}-\mathbf{2}\epsilon)^{|\mathcal{S}_n|}].*$$

**Definition:** The **energy spectrum**, $\mathcal{E}_n$, is defined by

$$\mathcal{E}_n(k) := \sum_{|S|=k} \hat{f}_n(S)^2, \quad k = 1, \ldots, n^2.$$

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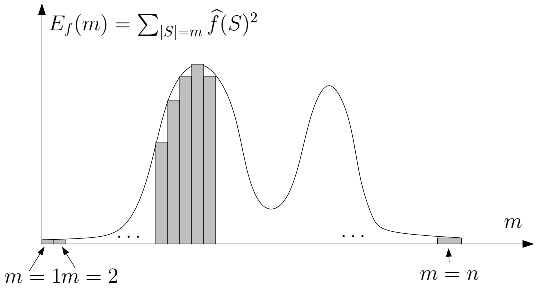
## Picture of the spectrum

The spectrum of a general function looks like this.



## The Fourier picture and noise sensitivity: continued

$$E(f_n(\omega)f_n(\omega^\epsilon)) = \sum_{S\subseteq H_n} \hat{f}_n(S)^2(1-2\epsilon)^{|S|}$$

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**Conclusion:** Noise sensitivity corresponds to $\mathcal{E}_f$ being concentrated on large $k$.

**Proposition (BKS, 1999):** $\{f_n\}$ is noise sensitive if and only if for every $k \geq 1$,

$$\lim_{n\to\infty} \sum_{|S|=k} \hat{f}_n(S)^2 = \lim_{n\to\infty} \mathcal{E}_{f_n}(k) = \lim_{n\to\infty} P(|\mathcal{S}_{f_n}| = k) = 0.$$

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## Quantitative noise sensitivity

**Quantitative** noise sensitivity results corresponds to knowing how **fast** the spectrum goes to $\infty$.

$$E(f(\omega)f(\omega^{\epsilon_n})) = E(f_n)^2 + \sum_{k=1}^{n^2} \mathcal{E}_n(k)(1-2\epsilon_n)^k.$$

Being sensitive to $(1/n)^\sigma$ is equivalent to vanishingly small spectrum below $n^\sigma$ (i.e., $\lim_{n\to\infty} \sum_{k=1}^{n^\sigma} \mathcal{E}_n(k) = 0$).

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## The spectrum for percolation

We "expect" that most of the spectrum is around $n^{3/4}$ because

- There was a heuristic involving the pivotals suggesting there was noise sensitivity with noise level $1/n^{3/4}$.
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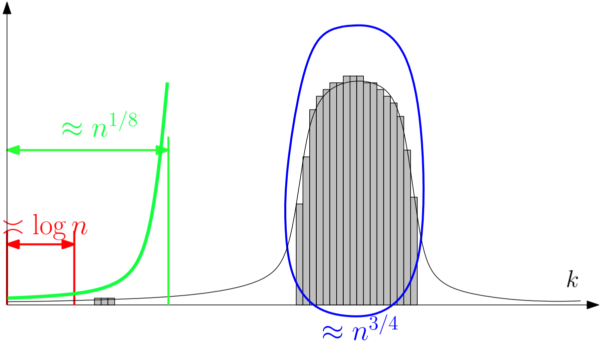

**The spectrum for percolation**

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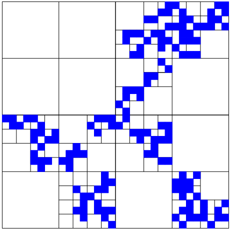
## The spectrum for percolation

To go all the way to $n^{3/4}$ (Garban, Pete and Schramm), one works with the whole random set $\mathcal{S}_n$.

A simple model where you have a discrete fractal and where one can "fairly easily" show that its typical behavior is described well by its expectation is **Fractal Percolation**.



## The dynamical percolation model

Dynamical Percolation was introduced in 1997 by Häggström, Peres and S. (HPS) (independently introduced by I. Benjamini)

Much to be said about this model (see a survey on my homepage) but we stick to critical percolation on the hexagonal lattice. Start at time 0 with an ordinary percolation realization and then let each hexagon evolve independently according to the 2-state continuous time Markov chain with

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## Dynamical percolation: Results

**Basic question:** Are there **exceptional** times at which the configuration **looks different** from ordinary percolation?

**Theorem. (Schramm-S., 2010):**

(i). For dynamical percolation on the hexagonal lattice, there are **exceptional times** at which percolation occurs.

(ii). The Hausdorff dimension of this fractal set of **exceptional times** belongs to $[1/6, 31/36]$.

The set of exceptional times is our **fourth fractal set**. It is sort of like the zero set of a Brownian motion.

**Garban, Pete and Schramm, 2010:** The Hausdorff dimension of the set of **exceptional times** is 31/36. On the square lattice, there are also **exceptional times**.

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## The second moment method: the key variable

The key variable is

$$X_n := \int_0^1 1_{V_{t,n}} \, dt$$

where $V_{t,n}$ is the event that at time $t$, there is an open path from the origin to distance $n$ away.

**So $X_n$ is the (Lebesgue) amount of time that the origin is connected to distant $n$ away.**

## The second moment method: the key step

**Key step:**

$$E(X_n^2) \leq O(1)E(X_n)^2$$

If true, Cauchy-Schwartz yields

$$\inf_n P(X_n > 0) > 0.$$

Hence

$$P(X_n > 0 \ \forall n) > 0$$

giving an **exceptional time.**

## The second moment method: bounding the correlation

To show

$$(*) E(X_n^2) \leq O(1) E(X_n)^2,$$

one needs good bounds on

$$P(V_{0,n} \cap V_{t,n}).$$

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**Two approaches**

Schramm-S. approach using a new connection with randomized algorithms in theoretical computer science yielded

$$(**)P(V_{0,n} \cap V_{t,n}) \leq O(1)t^{-5/6}P(V_{0,n})^2$$

The integrability of $t^{-5/6}$ yields (*) and the **"1/6 to spare"** gives via a "Frostman expected energy type argument" the lower bound.

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## Further reading

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