Infinite Ergodic Theory & Numbers

Bernd Otto Stratmann

Universität Bremen

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More details . . .

- M. Kesseböhmer & B.O. Stratmann.
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- Of particular interest will be the *even Stern–Brocot sequence*

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The end of the story (so far)

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For the even Stern–Brocot sequence we have that

$$\underset{n\to\infty}{*\text{-lim}} \quad \log(n^2) \sum_{p/q \in \mathcal{S}_n} q^{-2}\, \delta_{p/q} = \lambda,$$

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and for the Farey sequence we have that

$$\underset{n\to\infty}{*\text{-lim}} \quad \frac{\zeta(2)}{\log n} \sum_{p/q \in \mathcal{F}_n} q^{-2}\, \delta_{p/q} = \lambda.$$

The beginning of the story: Sum-Level Sets

- Recall the
  Regular continued fraction expansion

$$[a_1, a_2, \ldots] := \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ldots}},$$

where all the a_i are positive integers.

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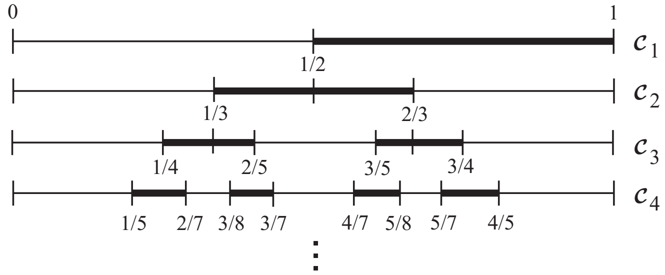
- $\limsup_n \mathcal{C}_n$ is equal to the set of all irrational numbers in $[0, 1]$.

A conjecture by Fiala & Kleban



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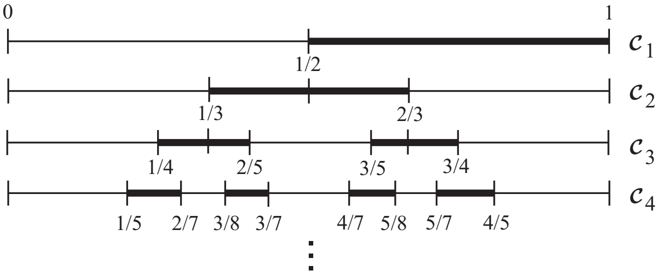

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$$\lambda(\mathcal{C}_1) = 1/2, \lambda(\mathcal{C}_2) = 1/3, \lambda(\mathcal{C}_3) = 3/10, \lambda(\mathcal{C}_4) = 39/140$$

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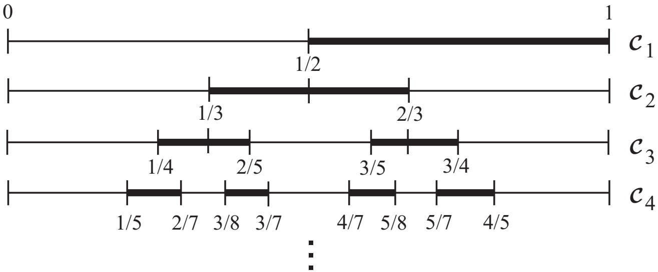

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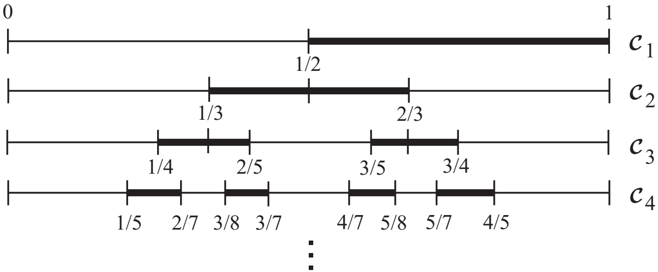

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$$\begin{array}{ll} \mathcal{C}_1 : & R \\ \mathcal{C}_2 : & LR \quad RR \\ \mathcal{C}_3 : & LLR \quad LRR \quad RLR \quad RRR \\ \mathcal{C}_4 : & LLLR \quad LLRR \quad LRRR \quad LRLR \quad RRLR \quad RRRR \quad RLRR \quad RLLR \\ & \vdots \end{array}$$

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The first proof of the Conjecture of Fiala & Kleban

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Theorem 1 (Kesseböhmer & Stratmann, 2009)

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Proposition (Kesseböhmer & Stratmann, 2010)

For each interval $[\alpha,\beta] \subset (0,1]$ *we have that*

$$\underset{n\to\infty}{*\text{-lim}} \quad \left(\frac{\log n}{\log(\beta/\alpha)} \cdot \lambda|_{T^{-n}([\alpha,\beta])}\right) = \lambda.$$

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For each rational number $v/w \in (0,1]$ we have that

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Elementary Lemma

For each rational number $v/w \in (0,1]$ we have that

$$\sum_{p/q \in T^{-n}(v/w)} \frac{1}{pq} = \frac{1}{vw}.$$

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Corollary

We have that

$$\sum_{\substack{\gamma \in \Gamma \\ d(0,\gamma(0)) \leq n}} e^{-d(0,\gamma(0))} \asymp \frac{n}{\log n} \quad \text{and} \quad \sum_{\substack{\gamma \in \Gamma \\ |\gamma| = n}} e^{-d(0,\gamma(0))} \asymp \frac{1}{\log n}.$$

A further direction: Kleinian groups

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Theorem

For a finitely generated, essentially free Kleinian group G with parabolic elements we have, for each $z, y \in \mathbb{H}$,

$$\sum_{g \in G, |g| \leq n} e^{-sd(z,g(w))} \asymp \begin{cases} n^{2\delta - r_{\max}} & \text{for} \quad \delta < (r_{\max}+1)/2 \\ n/\log n & \text{for} \quad \delta = (r_{\max}+1)/2 \\ n & \text{for} \quad \delta > (r_{\max}+1)/2. \end{cases}$$

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For the proof we refer to

- M. Kesseböhmer & B.O. Stratmann. 'A note on the algebraic growth rate of Poincaré series for Kleinian groups', preprint in arXiv:org (2010).

For the details we refer once more to . . .

- M. Kesseböhmer & B.O. Stratmann.
  'On the Lebesgue measure of sum-level sets for continued fractions',
  preprint in arXiv:org (2009);
  to appear in Discrete and Continuous Dynamical Systems 14 (2011).
- M. Kesseböhmer & B.O. Stratmann.
  'A dichotomy between uniform distributions of the Stern–Brocot and the Farey sequence',
  preprint in arXiv:org (2010).

Thanks for your attention!