Local homogeneity of measures (Part I)

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Joint work with Antti Käenmäki (Jyväskylä) and Tapio Rajala (Pisa)

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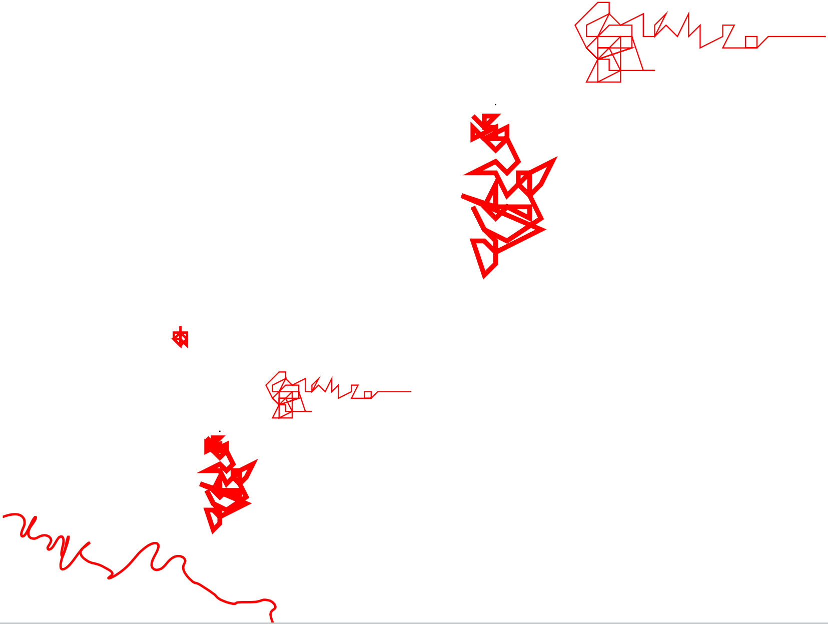

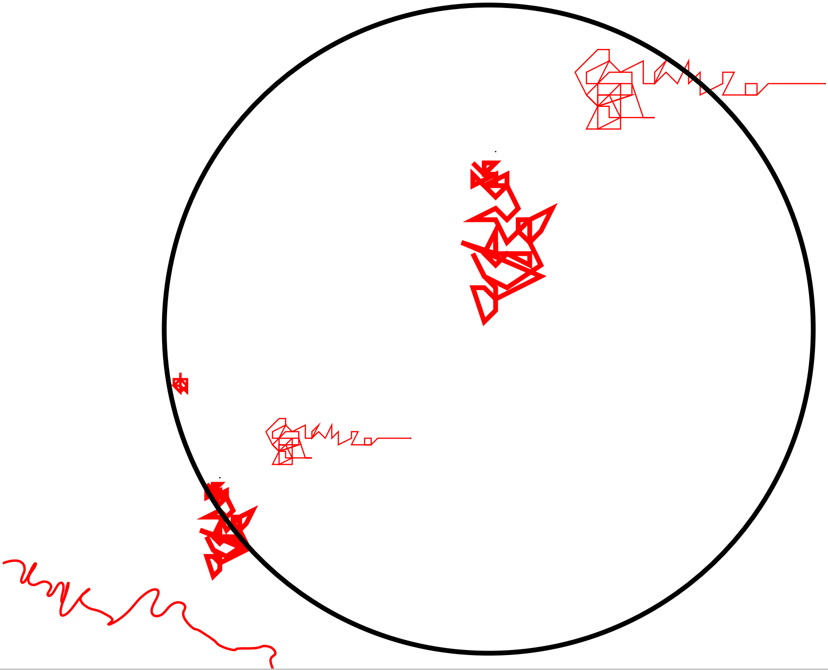

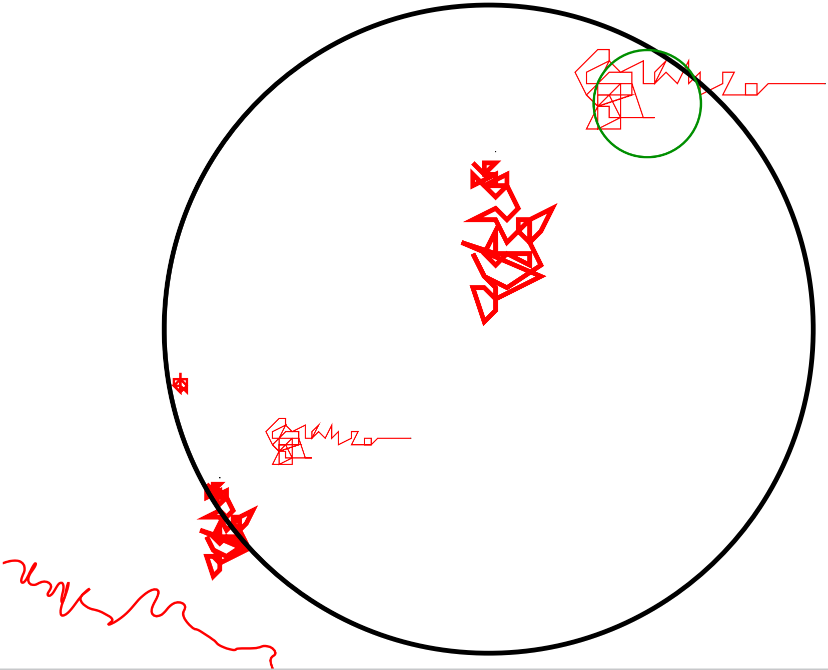

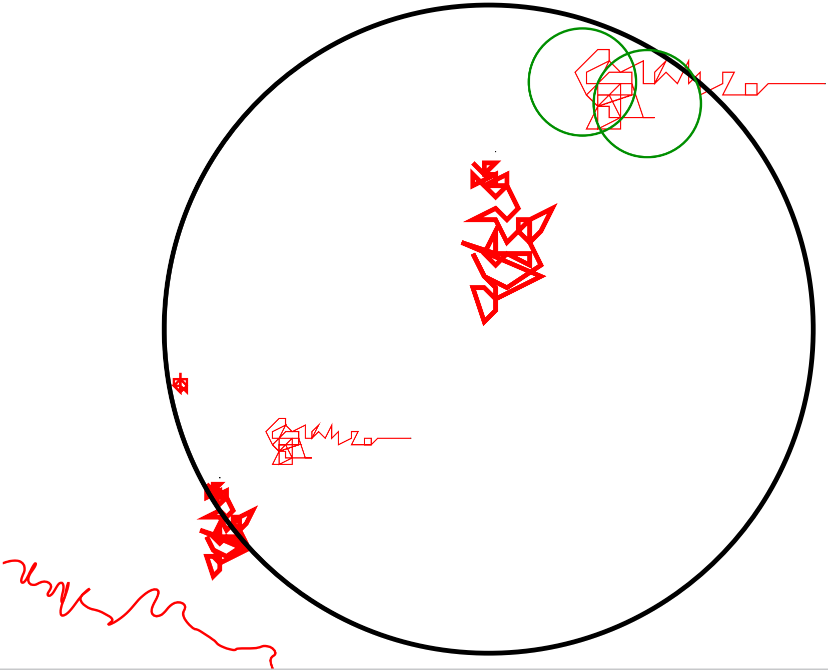

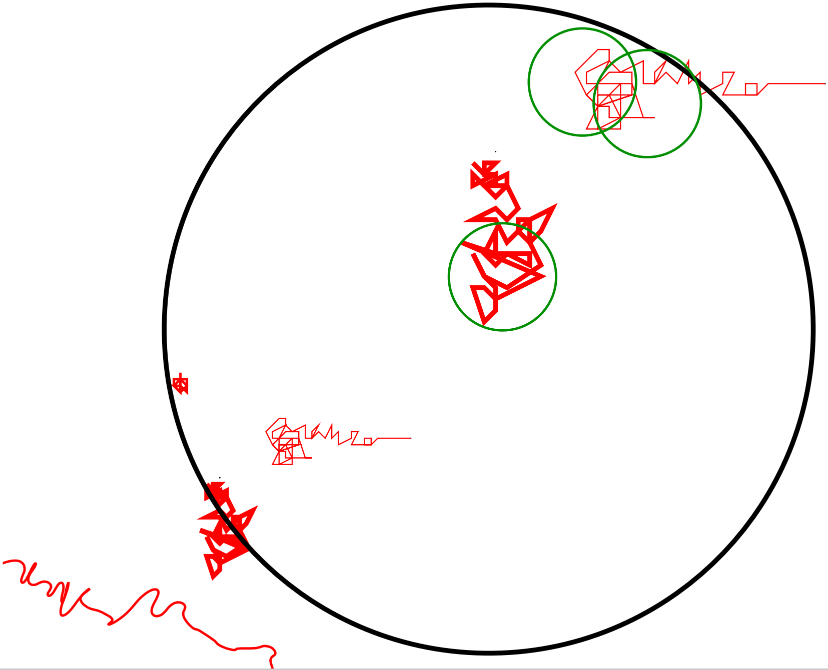

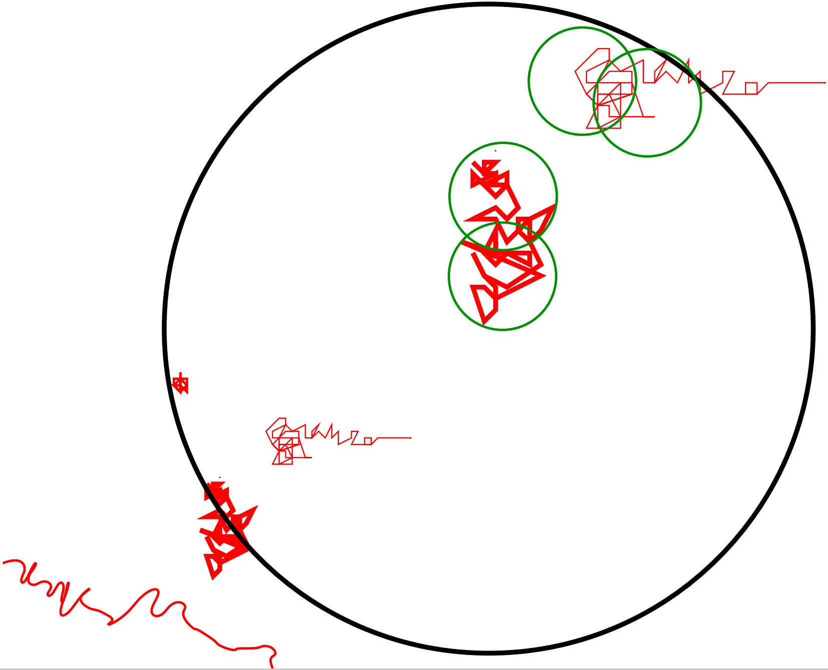

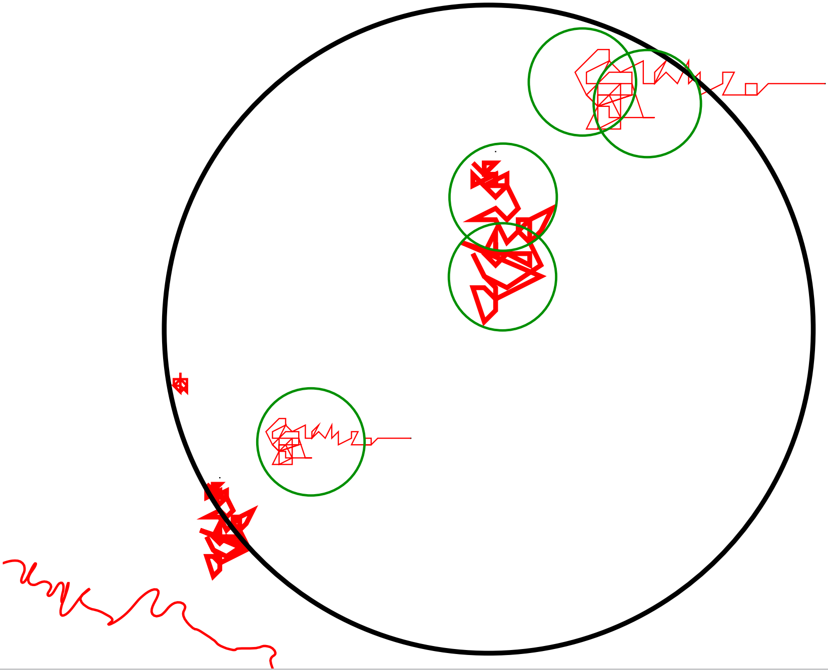

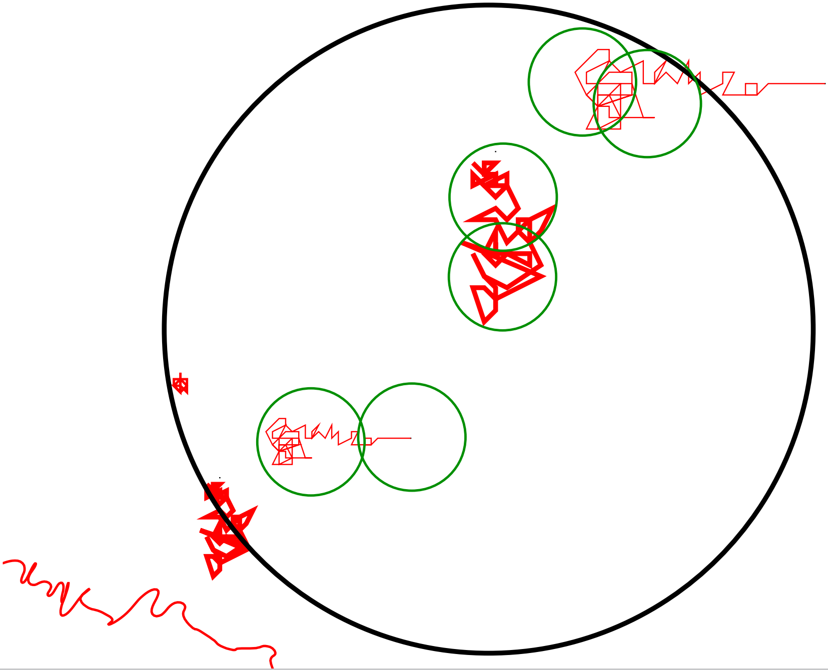

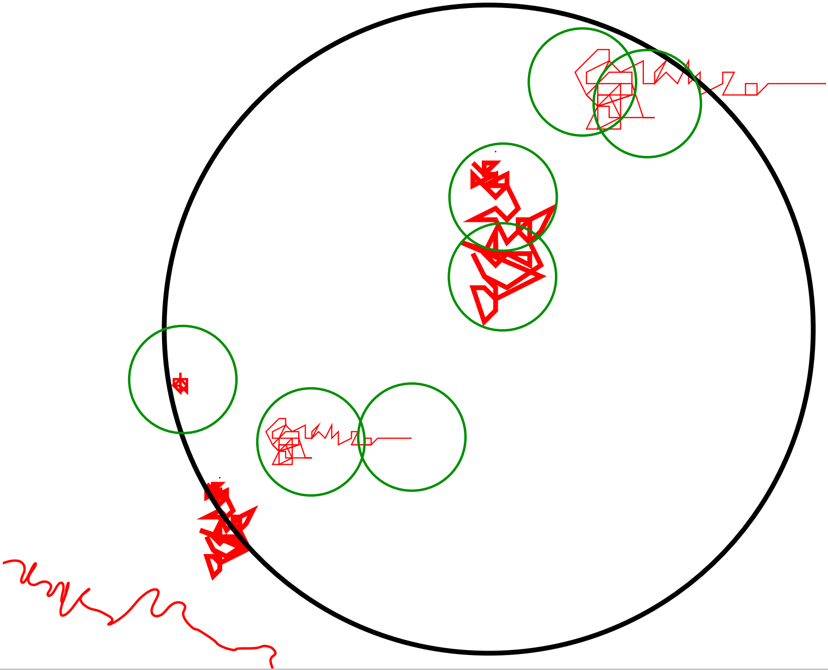

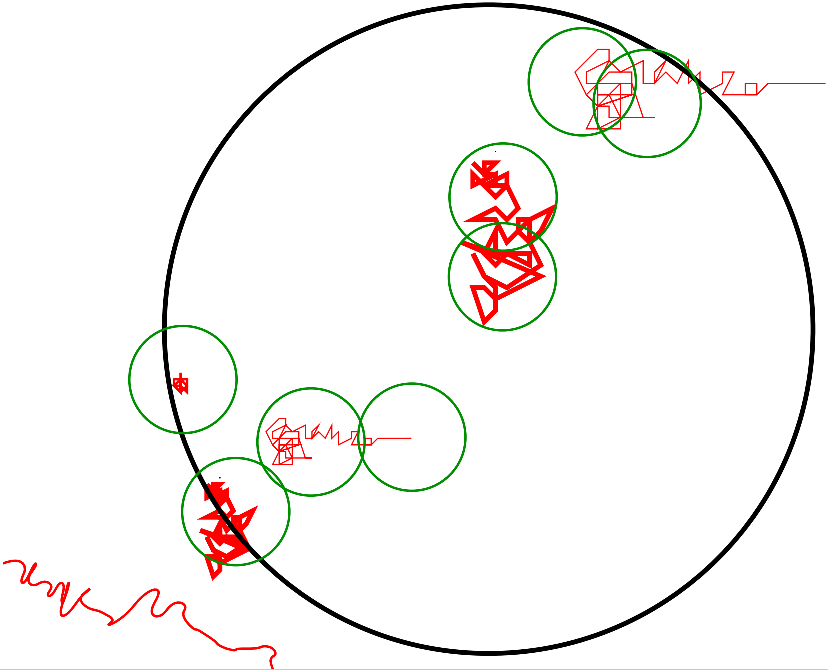

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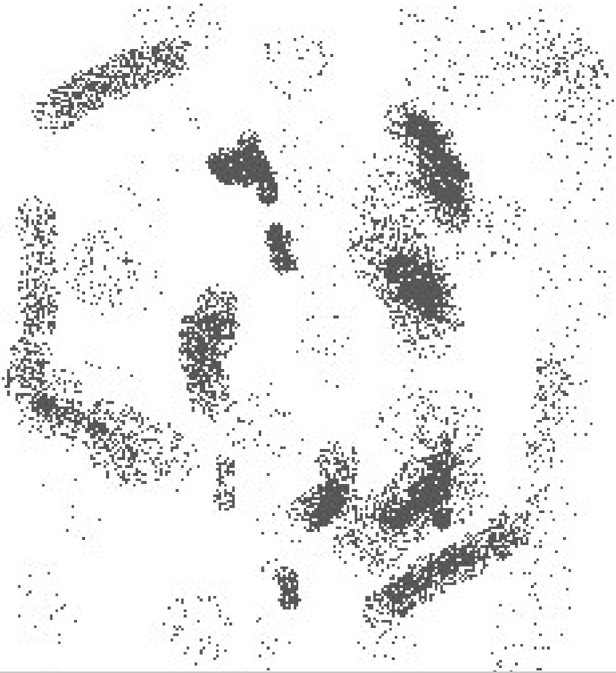

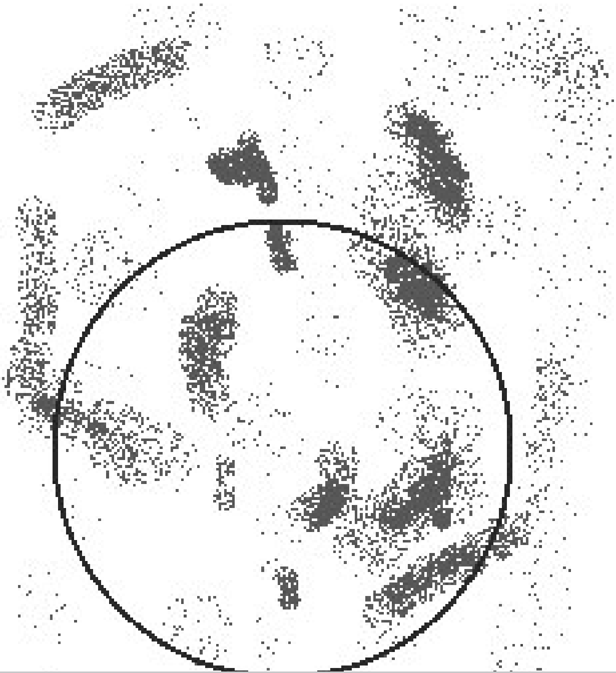

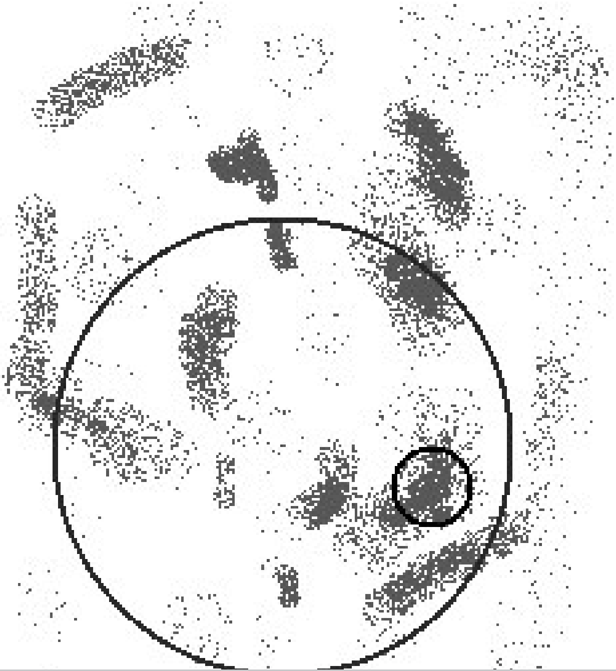

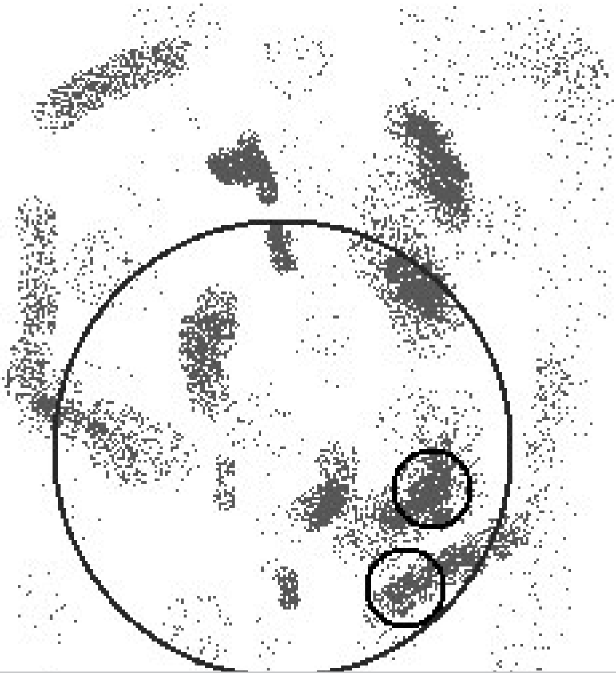

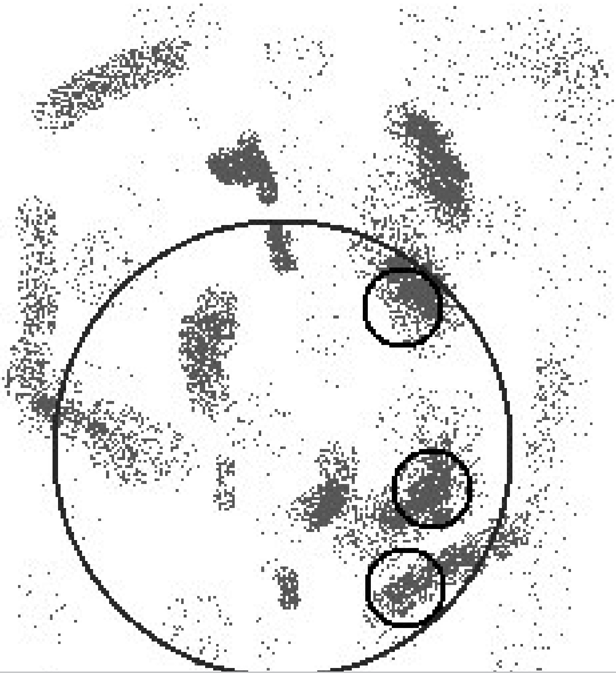

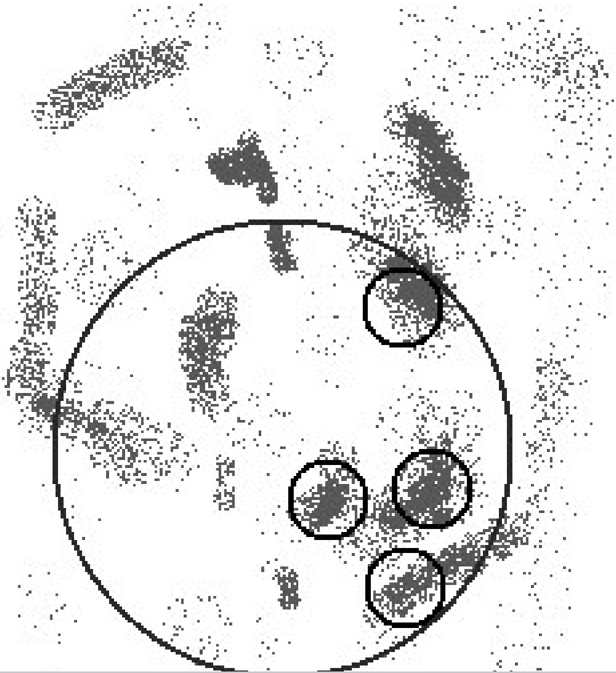

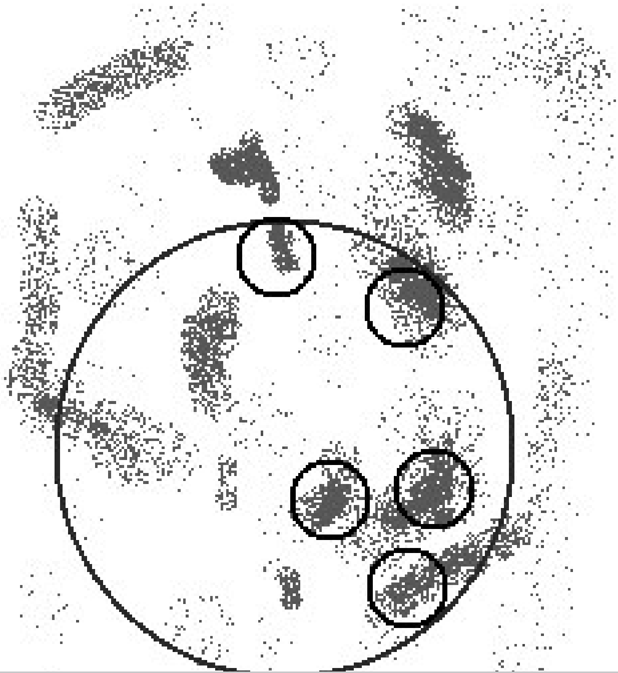

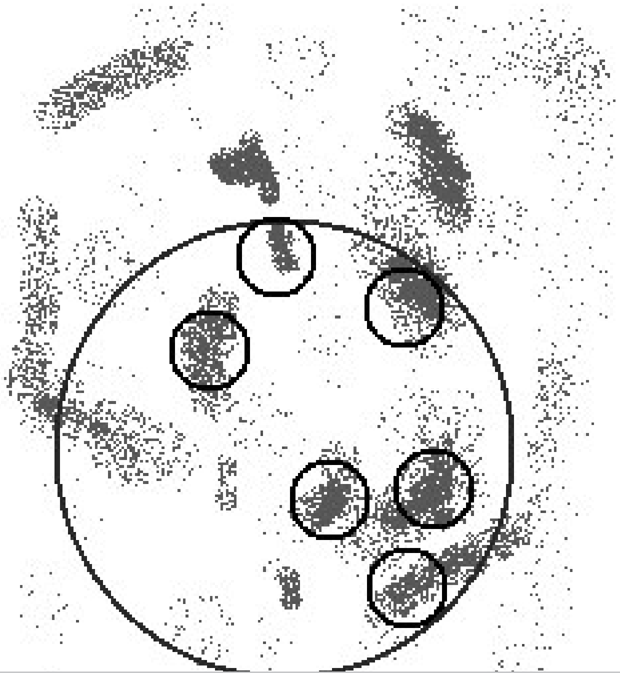

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Heuristic guess

If ε is small and $\dim \mu = s$ the answer should be roughly δ^{-s}.

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If μ is a measure on X and $x \in X$, we define the *upper and lower local dimensions of μ at x* by

$$\overline{\dim}_{\mathrm{loc}}(\mu, x) = \limsup_{r \downarrow 0} \frac{\log \mu\big(B(x, r)\big)}{\log r},$$

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A measure μ on X has the *density point property* if

$$\lim_{r \downarrow 0} \frac{\mu\big(A \cap B(x,r)\big)}{\mu\big(B(x,r)\big)} = 1$$

for μ-almost all $x \in A$ whenever $A \subset X$ is μ-measurable.

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$$\begin{aligned}\hom_{\delta,\varepsilon,r}(\mu, x) = \sup\{\#\mathcal{B} : \;& \mathcal{B} \text{ is a } (\delta r)\text{-packing of } B(x,r) \\ & \text{so that } \mu(B) > \varepsilon\mu\big(B(x,5r)\big) \text{ for all } B \in \mathcal{B}\}\end{aligned}$$

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$$\hom_\delta(\mu, x) = \lim_{\varepsilon\downarrow 0} \limsup_{r\downarrow 0} \hom_{\delta,\varepsilon,r}(\mu, x).$$

The *local homogeneity dimension of a measure μ at x* is then defined as

$$\dim_{\hom}(\mu, x) = \liminf_{\delta\downarrow 0} \frac{\log^+ \hom_\delta(\mu, x)}{-\log \delta},$$

where $\log^+(t) = \max\{0, \log t\}$.

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- $\dim_{\mathrm{hom}}(\mu, x)$ is roughly the least possible exponent s so that "large parts" of $B(x, r)$ in terms of μ can always be covered by δ^{-s} balls of radius δr for all small $r, \delta > 0$.

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- May be considered as a kind of local Assouad dimension for the measure μ around x.
- It is not essential that the constant in the definition of $\mathrm{hom}_{\delta,\varepsilon,r}$ is 5, but it is important that it is strictly greater than 1!

Main results

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Theorem A

If μ is a Radon measure on a doubling metric space X, then

$$\overline{\dim}_{\text{loc}}(\mu, x) \leq \dim_{\text{hom}}(\mu, x)$$

for μ-almost all $x \in X$.

A quantitative version

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Theorem B

Suppose X is a doubling metric space with a doubling constant N. If $0 < m < s$, then there exists a constant $\delta_0 = \delta_0(m, s, N) > 0$ such that for every $0 < \delta < \delta_0$ there is $\varepsilon_0 = \varepsilon_0(m, s, N, \delta) > 0$ so that for every Radon measure μ on X we have

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for all $0 < \varepsilon \leq \varepsilon_0$ and for μ-almost all $x \in X$ that satisfy $\overline{\dim}_{\mathrm{loc}}(\mu, x) > s$.

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Remark: Theorem B easily implies Theorem A.

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Key Lemma

Suppose μ is a Radon measure on a doubling metric space X with bounded support and $A \subset X$ is a Borel set with $\mu(A) > 0$. If there are $0 < \delta < 1$, $0 < t < s$, and $k_0 \in \mathbb{N}$ such that for every integer $k \geq k_0$ and for every δ^k-packing $\mathcal{B}$ of A there is an δ^{k-1}-packing $\mathcal{B}'$ of A so that

$$\delta^t \sum_{B \in \mathcal{B}} \mu(B)^{1-t/s} \leq \tfrac{1}{2} \sum_{B \in \mathcal{B}'} \mu(B)^{1-t/s},$$

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It is somewhat easier to prove such an estimate for dyadic mesh cubes of $\mathbb{R}^n$, or more generally for certain generalised nested cubes of a metric space X. One could also define the local homogeneity and prove results analogous to Theorems A and B with respect to such a grid structure. **However, it is essential for our applications that our concepts of local homogeneity and homogeneity dimension enjoy a "spherically symmetric" definition.**

Local L^q-spectrum and dimension

A (2^{-n})-partition of X is a partition of X into Borel sets Q such that for each such Q there is $x \in Q$ for which $B(x, 2^{-n}) \subset Q \subset B(x, C2^{-n})$.

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For each $n \in \mathbb{N}$, let $\mathcal{Q}_n$ be a (2^{-n})-partition of X and let $\mathcal{Q}_n(x,r) = \{Q \cap B(x,r) \,:\, Q \in \mathcal{Q}_n\}$. For a measure μ on X, $x \in X$ and $q \in \mathbb{R}$ ($q \geq 0$), we set

$$\tau_q(\mu, x) = \lim_{r \downarrow 0} \liminf_{n \to \infty} \frac{\log \sum_{Q \in \mathcal{Q}_n(x,r)} \mu(Q)^q}{\log 2^{-n}}.$$

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Remark

If we forget (x, r), then we arrive at the (classical) global L^q-spectrum and dimension, $\tau_q(\mu)$ and $\dim_q(\mu)$ (provided that $0 < \mu(X) < \infty$).

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Theorem C

If μ is a Radon measure on a doubling metric space X, then

$$\lim_{q\downarrow 1}\dim_q(\mu,x) \leq \underline{\dim}_{\mathrm{loc}}(\mu,x) \leq \overline{\dim}_{\mathrm{loc}}(\mu,x) \leq \lim_{q\uparrow 1}\dim_q(\mu,x)$$

for μ-almost all $x \in X$.

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for μ-almost all $x\in X$. In addition, if μ has the density point property, then

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Note

There is an analogous result for the global L^q-dimensions.

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Let μ be a Radon measure on a doubling metric space X with a compact support. Then

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for every $q \in \mathbb{R}$. In particular,

$$\dim_q(\mu) = \begin{cases} \max_{x \in \mathrm{spt}(\mu)} \dim_q(\mu, x), & \text{if } q < 1, \\ \min_{x \in \mathrm{spt}(\mu)} \dim_q(\mu, x), & \text{if } q > 1. \end{cases}$$