Quantitative recurrence properties in beta dynamical system and continued fraction system

Bao-Wei WANG

Université de Amiens

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Introduction

Background

- (X, d) a separable metric space ;
- $T : X \to X$ a map ;
- μ a finite T-invariant Borel measure.

- Poincaré Recurrence Thm :

$$\liminf_{n\to\infty} d(T^n x, x) = 0. \quad \mu - a.s.$$

- If T is ergodic, then for any $y_0 \in X$,

$$\liminf_{n\to\infty} d(T^n x, y_0) = 0. \quad \mu - a.s.$$

Questions : convergent speed

- Quantitative recurrence properties The possible convergence rate s.t.

$$\liminf_{n\to\infty} \underbrace{\bigcirc}_{???} d(T^n x, x) = 0$$

- shrinking target problem The possible convergence rate s.t.

$$\liminf_{n\to\infty} \underbrace{\bigcirc}_{???} d(T^n x, y_0) = 0$$

- Dynamical diophantine approximation Fix x_0, consider the set of y.

$$\liminf_{n\to\infty} \underbrace{\bigcirc}_{???} d(T^n x_0, y) = 0$$

(Bugeaud, Fan, Schmeling, Troubetzkoy, Liao, Seuret ...)

What are their sizes in $\rightarrow$ measure ? $\rightarrow$ Hausdorff dimension ?

Known Results : Size in measure

- Boshernitzan 1993. If the Hausdorff measure $\mathbb{H}^\alpha(X)$ is finite or σ-finte, then

$$\liminf_{n\to\infty} n^{1/\alpha} d(T^n x, x) < \infty, \quad \mu - a.s.$$

  Barreira & Saussol 2001. $X \subset \mathbb{R}^d$,

$$\liminf_{n\to\infty} n^{1/\alpha} d(T^n x, x) < \infty, \quad \mu - a.s.$$

  for any $\alpha > \underline{d}_\mu(x)$.

- Chernov & Kleinbock 2001. (Under some mixing conditions),

$$\liminf_{n\to\infty} n^{1/\alpha} d(T^n x, y_0) < \infty, \quad \mu - a.s.$$

  for any $\alpha > \underline{d}_\mu(y_0)$.

For more works on shrinking target problems : see **, Chazottes, Saussol, Fayad, Galatalo, J. Tseng, Kim, Fernàndez, Meliàn, Pestana,** $\cdots$

Size in dimension

Main concerns : Let (X, d) be a metric space, $T : X \to X$ a map. Find the dimension of the following dynamically defined limsup sets

- Quantitative recurrence :

$$R(T, \varphi) := \{x : d(T^n x, x) < \varphi(n), \text{i.o. } n \in \mathbb{N}\}$$

- Shrinking target problem :

$$S(T, y_0, \varphi) := \{x : d(T^n x, y_0) < \varphi(n), \text{i.o. } n \in \mathbb{N}\}$$

- Hill, Velani 1995, 1997. T an expanding rational map on Riemann sphere and J its Julia sets.

$$\dim_{\mathsf{H}} \left\{ x \in J : |T^n x - y_0| \leq e^{-S_n f(x)}, \text{ i. o.} \right\} = \inf\{s : P(-sf) \leq 0\}$$

where $P(-sf)$ is the pressure function with the potential $-sf$.

- Hill Velani 1999. (X, T), X n-dimensional torus and T a linear operator given by a matrix with integer coefficients.
- Urbański, 2002 Infinite conformal IFS.
- Stratmann and Urbański, 2002 Parabolic rational maps on Julia set.
- Fernàndez, Meliàn, Pestana 2007. General expanding Markov systems.

Two systems

- I β-expansion. For $\beta > 1$,

$$Tx = \beta x - \lfloor \beta x \rfloor.$$

Every $x \in [0, 1]$ can be expressed as

$$x = \frac{\epsilon_1(x, \beta)}{\beta} + \cdots + \frac{\epsilon_n(x, \beta)}{\beta^n} + \cdots .$$

- II Continued fractions. Gauss map : $T : [0, 1] \to [0, 1]$

$$T0 := 0,\ Tx = 1/x - \lfloor 1/x \rfloor,\ x \neq 0.$$

Then every irrational $x \in [0, 1]$ can be expressed uniquely as the continued fraction expansion with the form

$$x = \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \ddots}}.$$

Results

Results

Quantitative Recurrence

Theorem (Li-W-Wu-Xu, continued fraction)

Let $f : [0,1] \to \mathbb{R}^+$ *be a continuous function. Then*

$$\dim_H \left\{x \in [0,1] : |T^n x - x| < e^{f(x)+\cdots+f(T^{n-1}x)}, \quad \textit{i.o. } n \in \mathbb{N}\right\}$$
$$= \inf\{s : P(-s(\log|T'| + f)) \le 0\}.$$

Theorem (Tan-W, beta expansion)

$$\dim_H \left\{x \in [0,1] : |T^n x - x| < e^{f(x)+\cdots+f(T^{n-1}x)}, \quad \textit{i.o. } n \in \mathbb{N}\right\}$$

is given as the solution to $P(-s(\log|T'| + f)) = 0$.

Special feature or difficulties

- I. β-expansion. Length of a basic interval : $I_{n,\beta}(\epsilon_1, \cdots, \epsilon_n) := \{x : \epsilon_k(x, \beta) = \epsilon_k, 1 \leq k \leq n\}$.

  $$\left|I_{n,\beta}(\epsilon_1, \cdots, \epsilon_n)\right| \leq \beta^{-n}.$$

  When β is a Parry number (the expansion of 1 is finite), $\exists 0 < c < 1$ such that

  $$c\beta^{-n} \leq \left|I_{n,\beta}(\epsilon_1, \cdots, \epsilon_n)\right| \leq \beta^{-n}.$$

- II. continued fractions : Infinite conformal IFS.

Lower bound : focus on subsystems.

Main Bridge :

Lemma (Mauldin-Urbański, Conformal IFS, 1995)

Let $\{\phi_i : i \in I\}$ be an infinite conformal IFS. If the potential f has bounded total variation, then

$$P(f) = \sup\{P|_A(f) : A \text{ is a finite subset of } I\}$$

where $P|_A$ denote by the pressure defined on the subsystem generated by the finite IFS $\{\phi_i : i \in A\}$.

Lemma (Tan-W, beta expansion)

Let f be a continuous function on $[0,1]$.

$$P(T,f) = \sup\{P|_\Sigma(T,f) : \Sigma \text{ is } T \text{ invariant subsystem}\}.$$

Shrinking target

$$S(T, y_0, f) := \{x : |T^n x - y_0| < e^{f(x)+\cdots+f(T^{n-1}x)}, \text{ i.o. }\},$$

where f is positive and continuous.

Theorem (Shen-W, beta expansion ; Li-W-Wu-Xu, continued fractions)

- $\dim_H S(T, y_0, f)$ *is given by the solution to the pressure function*

$$P(T, -s(\log|T'| + f)) = 0.$$

A structure Lemma for beta expansion :

Lemma (Shen-W, beta expansion)

For any $\delta > 0$, for any ball $B(y_0, r)$, there exists a basic interval $I_n(\epsilon_1, \cdots, \epsilon_n)$ satisfying

- $I_n(\epsilon_1, \cdots, \epsilon_n) \subset B(y_0, r)$.
- $r^{1+\delta} \leq |I_n(\epsilon_1, \cdots, \epsilon_n)| < r.$

Final remark

Let $I_n = \{y \in X : T^n y = y\}$ or $I_n = \{y \in X : T^n y = y_0\}$.

Assume there exists $c_1 > c_2 > 0$ such that, for each $n \geq 1$

- Covering : $X \subset \bigcup_{y \in I_n} B\big(y, c_1 |(T^n)'(y)|^{-1}\big)$,
- Disjoint : $B\big(y, c_2 |(T^n)'(y)|^{-1}\big), y \in I_n$ disjoint.

Then possibly the dimension of the limsup set

$$\limsup_{n \to \infty} \bigcup_{y \in I_n} B\Big(y, e^{-(f(x) + \cdots + f(T^{n-1}x))}\Big)$$

should be given as $\inf\{s : P(-sf) \leq 0\}$.

• Compare it with the dimension of X :

$$\dim_{\mathsf{H}} X = \inf\{s : P(-s \log |T'|) \leq 0\}.$$

Thanks for your attention !